Supplementary file of the paper titled "3-D Image Denoising By Local Smoothing And Nonparametric Regression"

1 Proof of Theorem 3.1

We begin with two Lemmas. For simplicity, in the appendix, we use "$\sum$" to denote the sum over all design points $\{(x_i, y_j, z_k),\ i, j, k = 1, 2, \ldots, n\}$, unless otherwise mentioned.

Lemma 1 Under the conditions stated in Theorem 3.1, we have, for $i_1, i_2, i_3 = 0, 1, 2,$

$$\left\| \frac{1}{n^3 (h_n^*)^3} \sum \left(\frac{x_i - x}{h_n^*} \right)^{i_1} \left(\frac{y_j - y}{h_n^*} \right)^{i_2} \left(\frac{z_k - z}{h_n^*} \right)^{i_3} K \left(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*} \right) - \nu_{i_1 i_2 i_3} \right\|_{\Omega_{h_n^*}} = O\left(\frac{1}{n h_n^*} \right)$$

and

$$\left\| \frac{1}{n^3 h^3} \sum \varepsilon_{ijk} K \left(\frac{x_i - x}{h}, \frac{y_j - y}{h}, \frac{z_k - z}{h} \right) \right\|_{\Omega_{h_n^*}} = o\left(\frac{\beta_n \log(n)}{nh} \right) \ \ a.s.,$$

where $\nu_{i_1 i_2 i_3} = \int\int\int u^{i_1} v^{i_2} s^{i_3} K(u, v, s)\ dudvds$, for $i_1, i_2, i_3 = 0, 1, 2$. ■

Proof of Lemma 1 This is a straightforward generalization of Proposition 2 in Qiu (2009) from 2-D to 3-D cases. ■

Lemma 2 Under the conditions in Theorem 3.1, we have

$$\begin{aligned}
\|\widehat{a} - f\|_{\Omega_{\bar{J}, h_n^*}} &= O\left((h_n^*)^2\right) + o\left(\frac{\beta_n \log n}{n h_n^*} \right) \ \ a.s. \\
\|\widehat{b} - f'_x\|_{\Omega_{\bar{J}, h_n^*}} &= O\left(h_n^*\right) + o\left(\frac{\beta_n \log n}{n (h_n^*)^2} \right) \ \ a.s. \\
\|\widehat{c} - f'_y\|_{\Omega_{\bar{J}, h_n^*}} &= O\left(h_n^*\right) + o\left(\frac{\beta_n \log n}{n (h_n^*)^2} \right) \ \ a.s. \\
\|\widehat{d} - f'_z\|_{\Omega_{\bar{J}, h_n^*}} &= O\left(h_n^*\right) + o\left(\frac{\beta_n \log n}{n (h_n^*)^2} \right) \ \ a.s.
\end{aligned}$$

If $(x, y, z) \in J_{h_n^*} \backslash S_\epsilon$, then we have

$$\begin{pmatrix} \widehat{a}(x,y,z) \\ \widehat{b}(x,y,z) \\ \widehat{c}(x,y,z) \\ \widehat{d}(x,y,z) \end{pmatrix} = \begin{pmatrix} f_-(x,y,z) \\ f'_x(\tilde{x},\tilde{y},\tilde{z}) \\ f'_y(\tilde{x},\tilde{y},\tilde{z}) \\ f'_z(\tilde{x},\tilde{y},\tilde{z}) \end{pmatrix} + \begin{pmatrix} \phi_0(x,y,z)C(x,y,z) + O((h_n^*)^2) + o(\frac{\beta_n \log n}{n h_n^*}) \\ \phi_1(x,y,z)C(x,y,z) + \gamma_1(x,y,z)C_x(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ \phi_2(x,y,z)C(x,y,z) + \gamma_2(x,y,z)C_y(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ \phi_3(x,y,z)C(x,y,z) + \gamma_3(x,y,z)C_z(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \end{pmatrix} \quad a.s.$$

where $f_-(x, y, z)$ is the smaller one of the two one-sided (due to JLS) limits of f at (x, y, z), $(\tilde{x}, \tilde{y}, \tilde{z})$ is some point around (x, y, z) that satisfies (i) it is a continuity point of f that is on the same side of the JLS as (x, y, z), and (ii) $d_E((\tilde{x}, \tilde{y}, \tilde{z}), (x, y, z)) \sim O(1/n)$, $C(x, y, z)$, $C_x(x, y, z)$, $C_y(x, y, z)$, $C_z(x, y, z)$ are absolute jump magnitudes of $f(x, y, z)$ and its first order x, y and z partial derivatives, $\phi_1(x, y, z)$, $\phi_2(x, y, z)$ and $\phi_3(x, y, z)$ are three constants satisfying

$$\sqrt{\phi_1^2(x,y,z) + \phi_2^2(x,y,z) + \phi_3^2(x,y,z)} = O(1/h_n^*) \quad a.s.,$$

$\gamma_1(x, y, z)$, $\gamma_2(x, y, z)$ and $\gamma_3(x, y, z)$ are three constants between -1 and 1, and $\phi_0(x, y, z)$ is some constant between 0 and 1. ■

Proof of Lemma 2 When $(x, y, z) \in \Omega_{\bar{J}, h_n^*}$, by the Taylor's expansion, for any $(x_i, y_j, z_k) \in O^*(x, y, z)$, we have

$$\begin{aligned} \xi_{ijk} &= f(x_i, y_j, z_k) + \varepsilon_{ijk} \\ &= f(x,y,z) + (x_i - x)f'_x(x,y,z) + (y_j - y)f'_y(x,y,z) + (z_k - z)f'_z(x,y,z) \\ &+ O((h_n^*)^2) + \varepsilon_{ijk} \end{aligned}$$

So, we have

$$\begin{pmatrix} \sum \xi_{ijk} K_{ijk} \\ \sum \xi_{ijk}(x_i - x) K_{ijk} \\ \sum \xi_{ijk}(y_j - y) K_{ijk} \\ \sum \xi_{ijk}(z_k - z) K_{ijk} \end{pmatrix} = \begin{pmatrix} w_{000} & w_{100} & w_{010} & w_{001} \\ w_{100} & w_{200} & w_{110} & w_{101} \\ w_{010} & w_{110} & w_{020} & w_{011} \\ w_{001} & w_{101} & w_{011} & w_{002} \end{pmatrix} \begin{pmatrix} f(x,y,z) \\ f'_x(x,y,z) \\ f'_y(x,y,z) \\ f'_z(x,y,z) \end{pmatrix}$$

$$+ \begin{pmatrix} \sum O((h_n^*)^2) K_{ijk} + \sum \varepsilon_{ijk} K_{ijk} \\ \sum O((h_n^*)^2)(x_i - x) K_{ijk} + \sum \varepsilon_{ijk}(x_i - x) K_{ijk} \\ \sum O((h_n^*)^2)(y_j - y) K_{ijk} + \sum \varepsilon_{ijk}(y_j - y) K_{ijk} \\ \sum O((h_n^*)^2)(z_k - z) K_{ijk} + \sum \varepsilon_{ijk}(z_k - z) K_{ijk} \end{pmatrix}. \qquad (10)$$

By (3), (10), and *Lemma 1*, we have

$$\begin{pmatrix} \widehat{a}(x,y,z) \\ \widehat{b}(x,y,z) \\ \widehat{c}(x,y,z) \\ \widehat{d}(x,y,z) \end{pmatrix} = \begin{pmatrix} f(x,y,z) \\ f'_x(x,y,z) \\ f'_y(x,y,z) \\ f'_z(x,y,z) \end{pmatrix} + \begin{pmatrix} O((h_n^*)^2) + o(\frac{\beta_n \log n}{n h_n^*}) \\ O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \end{pmatrix} a.s. \qquad (11)$$

Under the conditions in Theorem 3.1, it is clear that (11) is uniformly true for $(x,y,z) \in \Omega_{\bar{J}, h_n^*}$. This an easy application of Lemma 1.

Now, if $(x,y,z) \in J_{h_n^*} \backslash S_\epsilon$ and n is large enough so that $h_n^* < \epsilon$, then $O^*(x,y,z)$ is divided into two parts I_1 and I_2 by the JLS. Without loss of generality, let us assume that there is a positive jump from I_1 to I_2 at (x,y,z). Then, when $(x_i, y_j, z_k) \in I_1$, we have

$$\begin{aligned} \xi_{ijk} &= f(x_i, y_j, z_k) + \varepsilon_{ijk} \\ &= f_-(x,y,z) + (x_i - x) f'_x(\tilde{x}, \tilde{y}, \tilde{z}) + (y_j - y) f'_y(\tilde{x}, \tilde{y}, \tilde{z}) + (z_k - z) f'_z(\tilde{x}, \tilde{y}, \tilde{z}) \\ &+ O((h_n^*)^2) + \varepsilon_{ijk} \end{aligned}$$

Similarly, when $(x_i, y_j, z_k) \in I_2$, we have

$$
\begin{aligned}
\xi_{ijk} &= f(x_i, y_j, z_k) + \varepsilon_{ijk} \\
&= f_-(x,y,z) + (x_i - x) f'_x(\tilde{x}, \tilde{y}, \tilde{z}) + (y_j - y) f'_y(\tilde{x}, \tilde{y}, \tilde{z}) + (z_k - z) f'_z(\tilde{x}, \tilde{y}, \tilde{z}) \\
&+ C(x,y,z) + (x_i - x) C_x(x,y,z) + (y_j - y) C_y(x,y,z) + (z_k - z) C_z(x,y,z) \\
&+ O((h_n^*)^2) + \varepsilon_{ijk}
\end{aligned}
$$

where $(\tilde{x}, \tilde{y}, \tilde{z})$ is some point in I_1 that satisfies the conditions stated in Lemma 2. By (3) and the above two expressions, we have

$$
\begin{pmatrix} \widehat{a}(x,y,z) \\ \widehat{b}(x,y,z) \\ \widehat{c}(x,y,z) \\ \widehat{d}(x,y,z) \end{pmatrix} = \begin{pmatrix} f_-(x,y,z) \\ f'_x(\tilde{x}, \tilde{y}, \tilde{z}) \\ f'_y(\tilde{x}, \tilde{y}, \tilde{z}) \\ f'_z(\tilde{x}, \tilde{y}, \tilde{z}) \end{pmatrix} + \begin{pmatrix} \phi_0(x,y,z) C(x,y,z) + O((h_n^*)^2) + o(\frac{\beta_n \log n}{n h_n^*}) \\ \phi_1(x,y,z) C(x,y,z) + \gamma_1(x,y,z) C_x(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ \phi_2(x,y,z) C(x,y,z) + \gamma_2(x,y,z) C_y(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \\ \phi_3(x,y,z) C(x,y,z) + \gamma_3(x,y,z) C_z(x,y,z) + O(h_n^*) + o(\frac{\beta_n \log n}{n (h_n^*)^2}) \end{pmatrix} a.s.
$$

where

$$
\begin{aligned}
\phi_0(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\phi_1(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (x_i - x) K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (x_i - x)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\phi_2(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (y_j - y) K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (y_j - y)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\phi_3(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (z_k - z) K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (z_k - z)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\gamma_1(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (x_i - x)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (x_i - x)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\gamma_2(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (y_j - y)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (y_j - y)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})} \\
\gamma_3(x,y,z) &= \frac{\sum_{(x_i,y_j,z_k) \in I_2} (z_k - z)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}{\sum (z_k - z)^2 K(\frac{x_i - x}{h_n^*}, \frac{y_j - y}{h_n^*}, \frac{z_k - z}{h_n^*})}.
\end{aligned}
$$

From the above expressions, it is obvious that $\gamma_1(x,y,z)$, $\gamma_2(x,y,z)$ and $\gamma_3(x,y,z)$ are constants between 0 and 1, and $\phi_0(x,y,z)$ is a constant between 0 and 1. Without loss of generality, let $C_x(x,y,z)$, $C_y(x,y,z)$ and $C_z(x,y,z)$ denote absolute jump magnitudes of f'_x, f'_y and f'_z, then $\gamma_1(x,y,z)$, $\gamma_2(x,y,z)$ and $\gamma_3(x,y,z)$ are constants between -1 and 1. By similar arguments to those in Qiu and Yandell (1997) it is not difficult to check that $\sqrt{\phi_1(x,y,z)^2+\phi_2(x,y,z)^2+\phi_3(x,y,z)^2} = O(1/h_n^*)$ a.s. So, Lemma 2 is proved. ■

Proof of Theorem 3.1 For a design point $(x,y,z) \in \Omega_{\bar{S},\epsilon}$, if it is more than h_n^* away from any JLS, then at least one of $O^*(x_{N_1}, y_{N_1}, z_{N_1})$ and $O^*(x_{N_2}, y_{N_2}, z_{N_2})$ is located in a same continuous region as (x,y,z). So, we have

$$\delta(x,y,z) \leq \|\widehat{\boldsymbol{\beta}}(x,y,z) - \widehat{\boldsymbol{\beta}}_{N_1}(x,y,z)\| = O\left(h_n^*\right) + o\left(\frac{\beta_n \log n}{n(h_n^*)^2}\right) \quad a.s.$$

The above expression is a direct conclusion of Lemma 2. Using the fact that $\chi^2_{3,\alpha_n} = O(-\log\alpha_n)$, the expression (8) and Lemma 1, it is not difficult to check that the threshold value $u_n = O(\frac{n\sqrt{-\log\alpha_n}}{(nh_n^*)^{5/2}})$ a.s. The fact that $\chi^2_{3,\alpha_n} = O(-\log\alpha_n)$ can be proved easily by using $\chi^2_{3,\alpha_n} \leq 3\chi^2_{1,\alpha_n/3}$ and the Mill's inequality regarding normal tail probabilities. So, under the condition that $\frac{(nh_n^*)^{7/2}}{n^2\sqrt{-\log\alpha_n}} = o(1)$, we have $\frac{\delta(x,y,z)}{u_n} = o(1)$ a.s. resulting $\delta(x,y,z) < u_n$ a.s. (i.e., (x,y,z) is not detected as an edge point) when n is large enough, and this is uniformly true for all $(x,y,z) \in \Omega_{\bar{S},\epsilon} \cap \Omega_{\bar{J},h_n^*}$. Therefore,

$$\sup_{(x,y,z)\in\widehat{D}_n \bigcap \Omega_{\bar{S},\epsilon}} \; \inf_{(x',y',z')\in D \bigcap \Omega_{\bar{S},\epsilon}} d_E((x,y,z)^T, (x',y',z')^T) = O(h_n^*) \quad a.s. \tag{12}$$

On the other hand, if (x,y,z) is a non-singular point on a JLS, then by Lemma 2, we have

$$\delta(x,y,z) \sim C(x,y,z)\sqrt{\phi_1(x,y,z)^2+\phi_2(x,y,z)^2+\phi_3(x,y,z)^2} + O(h_n^*) + o(\frac{\beta_n \log n}{n(h_n^*)^2}) \; a.s.$$

Since $\sqrt{\phi_1(x,y,z)^2 + \phi_2(x,y,z)^2 + \phi_3(x,y,z)^2} = O(1/h_n^*)$ a.s., by the the condition that $\sqrt{-\log \alpha_n/(nh_n^*)^3} = o(1)$, we have $\delta(x,y,z) > u_n$ a.s. So, (x,y,z) would be detected as an edge pixel when n is large enough. Since $\min_{(x,y,z)\in D \bigcap \Omega_{\bar{S},\epsilon}} C(x,y,z) > 0$ (see the definition of *singular points* in Section 3), the above result is uniformly true for $(x,y,z) \in D \bigcap \Omega_{\bar{S},\epsilon}$. Therefore,

$$\sup_{(x,y,z)\in D \bigcap \Omega_{\bar{S},\epsilon}} \inf_{(x',y',z')\in \widehat{D}_n \bigcap \Omega_{\bar{S},\epsilon}} d_E((x,y,z)^T, (x',y',z')^T) = O(h_n^*) \;\; a.s. \tag{13}$$

By (12) and (13), Theorem 3.1 is proved. ■

2 Proof of Theorem 3.2

Lemma 3 Besides the conditions in Theorem 3.1, let us further assume that $(x,y,z) \in J_{h_n} \backslash S_\epsilon$, the JLS has unique tangent plane at (x_*, y_*, z_*), the point on D that is closest to (x,y,z), and the bandwidth h_n satisfies the conditions that $h_n = o(1)$, $1/(nh_n) = o(1)$, $h_n^*/h_n = o(1)$, and $\breve{h}_n = ch_n$ where $c > 0$ is a constant. Then, the local plane fitted by the algorithm in Section 2.2.1 converges almost surely to the tangent plane of the JLS at (x_*, y_*, z_*) both in normal direction and pointwise as $n \to \infty$. ■

Proof of Lemma 3 Assume that the normal direction of the tangent plane of the JLS at (x_*, y_*, z_*) is $(\rho_{x_*}, \rho_{y_*}, \rho_{z_*})^T$ with $\sqrt{\rho_{x_*}^2 + \rho_{y_*}^2 + \rho_{z_*}^2} = 1$. Without loss of generality, we further assume that $(\rho_{x_*}, \rho_{y_*}, \rho_{z_*})^T = (0,0,1)^T$ and that n is large enough so that $h_n < \epsilon$. So, the equation of the tangent plane of the JLS at (x_*, y_*, z_*) is $z = z_*$ and any nonsingular point on the JLS in $O(x,y,z)$ satisfies $z = z_* + O(h_n^2)$. Therefore, the gradient direction at any point on JLS in $O(x,y,z)$ can be written as $(O(h_n^2), O(h_n^2), 1 + O(h_n^2))^T$. Now, if

$(x^*, y^*, z^*) \in \left(J_{h_n^*} \backslash S_\epsilon\right) \cap O(x, y, z)$, then

$$\widehat{\boldsymbol{\beta}}^*(x^*, y^*, z^*) = \left(\frac{\widehat{b}(x^*, y^*, z^*)}{\|\widehat{\boldsymbol{\beta}}(x^*, y^*, z^*)\|}, \frac{\widehat{c}(x^*, y^*, z^*)}{\|\widehat{\boldsymbol{\beta}}(x^*, y^*, z^*)\|}, \frac{\widehat{d}(x^*, y^*, z^*)}{\|\widehat{\boldsymbol{\beta}}(x^*, y^*, z^*)\|} \right)^T.$$

The expressions of $\widehat{b}(x^*, y^*, z^*)$, $\widehat{c}(x^*, y^*, z^*)$, $\widehat{d}(x^*, y^*, z^*)$ and $\widehat{\boldsymbol{\beta}}(x^*, y^*, z^*)$ can be obtained from Lemma 2. From Lemma 1, it is easy to check that $\phi_3(x^*, y^*, z^*) \sim \frac{O(n^3 (h_n^*)^4)}{O(n^3 (h_n^*)^5)} = O(\frac{1}{h_n^*})$ a.s. Since the gradient direction at any point of the JLS in $O(x, y, z)$ can be written as $(O(h_n^2), O(h_n^2), 1 + O(h_n^2))^T$, from the expressions of $\phi_1(x^*, y^*, z^*)$ and $\phi_2(x^*, y^*, z^*)$ in Lemma 2, we can see that both of them are of the order $\frac{O(n^3 (h_n^*)^3 h_n^2 h_n^*)}{O(n^3 (h_n^*)^5)} = O\left(\frac{h_n^2}{h_n^*}\right)$ a.s. Then, we have $\widehat{\boldsymbol{\beta}}^*(x^*, y^*, z^*) = (O(h_n^2), O(h_n^2), 1+O(h_n^2))^T$ a.s. So, the matrix $G = (g_{i_1,i_2}, i_1, i_2 = 1, 2, 3)$ defined in (5) has the property that

$$\begin{aligned} g_{i_1,i_2} &= O(h_n^2) \;\; a.s., \text{ if } (i_1, i_2) \neq (3, 3), \\ g_{3,3} &= 1 + O(h_n^2) \;\; a.s. \end{aligned}$$

Since G is a real symmetric matrix, the eigenvalues of G are $\tau + 2\sqrt{p}\cos\gamma$, $\tau - \sqrt{p}(\cos\gamma + \sqrt{3}\sin\gamma)$ and $\tau - \sqrt{p}(\cos\gamma - \sqrt{3}\sin\gamma)$, where $3\tau = trace(G)$, $2q = det(G - \tau I)$, $6p$ is the sum of squares of the elements of $(G - \tau I)$, and $\gamma = \frac{1}{3}\tan^{-1}\frac{\sqrt{p^3 - q^2}}{q}$ with $0 \leq \gamma \leq \pi$. Therefore, we have, $\tau = \frac{1}{3} + O(h_n^2)$ a.s., $6p = \frac{2}{3} + O(h_n^2)$ a.s., $q = \frac{1}{27} + O(h_n^2)$ a.s. and $\gamma = O(h_n^2)$ a.s. and thus the three eigenvalues of G are of the orders $1 + O(h_n)$, $O(h_n)$, and $O(h_n)$, a.s., respectively. The eigenvector corresponding to a eigenvalue λ can be found by finding the solution for $\mathbf{e} = (e_1, e_2, e_3)^T$ from the two equations $G\mathbf{e} = \lambda\mathbf{e}$ and $\mathbf{e}'\mathbf{e} = 1$. If

$\lambda = \lambda_1$ with λ_1 being the largest eigenvalue of G, we have

$$\begin{aligned} e_1 &= \eta \left(\frac{g_{1,2}g_{2,3} - g_{1,3}(g_{2,2} - \lambda_1)}{(g_{1,1} - \lambda_1)(g_{2,2} - \lambda_1) - g_{1,2}g_{2,1}} \right), \\ e_2 &= \eta \left(\frac{g_{1,3}g_{2,1} - g_{2,3}(g_{1,1} - \lambda_1)}{(g_{1,1} - \lambda_1)(g_{2,2} - \lambda_1) - g_{1,2}g_{2,1}} \right), \\ e_3 &= \eta, \end{aligned}$$

where η is such that $e_1^2 + e_2^2 + e_3^2 = 1$. After combining this result and the results about the elements of G and λ_1, we have $e_1 = O(h_n)$ a.s., $e_2 = O(h_n)$ a.s. and $e_3 = 1 + O(h_n)$ a.s., from which we have $\mathbf{e}$ converges to $(0, 0, 1)^T$ a.s., as $n \to \infty$. Therefore, the normal direction of the fitted plane by the algorithm in Section 2.2.1 converges to the normal direction of the JLS at (x_*, y_*, z_*) almost surely. From Theorem 3.1, it is clear that the center of $\widehat{D}_n \cap O(x, y, z)$ converges almost surely to some point on the tangent plane of the JLS at (x_*, y_*, z_*), and the convergence rate is $O(h_n)$. This concludes the proof of *Lemma 3*.

Lemma 4 Besides the conditions in Theorem 3.1, let us further assume that $(x, y, z) \in J_{h_n} \backslash S_\epsilon$, the JLS has two different one-sided tangent planes at some point $(x_*, y_*, z_*) \in O(x, y, z)$, and the bandwidth h_n satisfies the conditions that $h_n = o(1)$, $1/(nh_n) = o(1)$, $h_n^*/h_n^3 = o(1)$, and $\breve{h}_n = ch_n$ where $c > 0$ is a constant. Then, the two planes fitted by the algorithm in Section 2.2.2 converges almost surely to the two one-sided tangent planes of the JLS at (x_*, y_*, z_*).

Proof of Lemma 4 Without loss of generality we can assume that two different one-sided tangent planes at (x_*, y_*, z_*) are $x = x_*$ and $y - y_* = \kappa(x - x_*)$ where κ is a constant, which are labeled P_1 and P_2 respectively. They intersect at a straight line L and clearly $(x_*, y_*, z_*) \in L$. Let us consider a plane S that passes L and separates P_1 and P_2 in $O(x, y, z)$. Then, S divides $O(x, y, z)$ into two parts $N_1(x, y, z)$ and $N_2(x, y, z)$, and it divides $D \cap O(x, y, z)$ into two parts $D \cap N_1(x, y, z)$ and $D \cap N_2(x, y, z)$. From the proof of *Lemma 3*, we see that if a point (x^*, y^*, z^*) is more than h_n^* away from L and $(x^*, y^*, z^*) \in (J_{h_n} \backslash S_\epsilon) \cap N_1(x, y, z)$, then $\widehat{\boldsymbol{\beta}}^*(x^*, y^*, z^*) = (1 + O(h_n^2), O(h_n^2), O(h_n^2))^T$ a.s., which converges to the normal direction of P_1 as $n \to \infty$. Likewise, if (x^*, y^*, z^*) is more than h_n^* away from L and $(x^*, y^*, z^*) \in (J_{h_n} \backslash S_\epsilon) \cap N_2(x, y, z)$, then $\widehat{\boldsymbol{\beta}}^*(x^*, y^*, z^*) = \left(\frac{\kappa + O(h_n^2)}{\sqrt{1+\kappa^2}}, \frac{1+O(h_n^2)}{\sqrt{1+\kappa^2}}, O(h_n^2) \right)^T$ a.s. which converges to the normal direction of P_2 as $n \to \infty$. Define,

$$
\begin{aligned}
m_1 &= \text{number of elements in } \widehat{D}_n \cap N_1(x, y, z) \text{ that are more than } h_n^* \text{ away from } L. \\
m_2 &= \text{number of elements in } \widehat{D}_n \cap N_2(x, y, z) \text{ that are more than } h_n^* \text{ away from } L. \\
m_3 &= \text{number of elements in } \widehat{D}_n \cap O(x, y, z) \text{ that are at most } h_n^* \text{ away from } L.
\end{aligned}
$$

Since m is the number of points in $\widehat{D}_n \cap O(x, y, z)$ and $\frac{h_n^*}{h_n^3} = o(1)$, we have $\frac{m_3}{m} = O\left(\frac{h_n^*}{h_n}\right) = o(h_n^2)$ a.s. Similarly we can show that $\frac{m_1}{m}$ and $\frac{m_2}{m}$ are strictly between 0 and 1 when n is large enough. Then the matrix G has the properties that

$$
\begin{aligned}
g_{1,1} &= \frac{1}{m}\left(m_1 + \frac{\kappa^2 m_2}{1+\kappa^2} + m_3\zeta\right) + O(h_n^2) \ \ a.s. \\
g_{1,2} &= g_{2,1} = \frac{1}{m}\left(\frac{\kappa^2 m_2}{1+\kappa^2} + m_3\zeta\right) + O(h_n^2) \ \ a.s. \\
g_{2,2} &= \frac{1}{m}\left(\frac{m_2}{1+\kappa^2} + m_3\zeta\right) + O(h_n^2) \ \ a.s. \\
g_{3,3} &= \frac{m_3\zeta}{m} + O(h_n^2) \ \ a.s. \\
g_{1,3} &= g_{3,1} = \frac{m_3\zeta}{m} + O(h_n^2) \ \ a.s. \\
g_{2,3} &= g_{3,2} = \frac{m_3\zeta}{m} + O(h_n^2) \ \ a.s.
\end{aligned}
\tag{14}
$$

where ζ is a number between -1 and 1. Consequently, when n is sufficiently large $|g_{1,1}|$, $|g_{1,2}|$, $|g_{2,2}| > 0$ a.s. and $|g_{1,3}|$, $|g_{2,3}|$ and $|g_{3,3}|$ are all of the order $o(1)$ a.s. If λ is an eigenvalue of G, then we have $\det(G - \lambda I) = 0$. Combining this with (14), we have $(g_{3,3} - \lambda)\left((g_{1,1} - \lambda)(g_{2,2} - \lambda) - g_{1,2}g_{2,1}\right) = O(h_n^2)$ a.s. Therefore, when n is large enough, one solution of λ is $O(h_n^2)$ a.s. and the other two solutions differ from 0 by at least a non-zero constant because of the Cauchy-Schwarz inequality that $|g_{1,1}g_{2,2} - g_{1,2}g_{2,1}| > 0$. Now, proceeding similarly as in the proof of *Lemma 3*, we can check that the eigenvector corresponding to the smallest eigenvalue of G is $(O(h_n^2), O(h_n^2), 1 + O(h_n^2))^T$ a.s., which converges to $(0, 0, 1)^T$, the direction of L.

Now, we can check that

$$\overline{\boldsymbol{\beta}}^* = \left(\frac{m_1}{m} + \frac{m_2\kappa}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2), \frac{m_2}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2), O(h_n^2) + o(h_n^2)\right)^T \ a.s.$$

So, the orthogonal direction of the plane P defined in Section 2.2.2 is

$$\vec{\boldsymbol{l}} = \left(-\frac{m_2}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2), \frac{m_1}{m} + \frac{m_2\kappa}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2), O(h_n^2) + o(h_n^2)\right)^T \ a.s.$$

The inner product of this orthogonal direction with $\widehat{\boldsymbol{\beta}}(x^*, y^*, z^*)$ is

$$\begin{cases} -\frac{m_2}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2) \ \ a.s., & \text{if } (x^*, y^*, z^*) \in \widehat{D}_n \cap N_1(x, y, z) \text{ and is more than } h_n^* \text{ away from } L. \\ \frac{m_1}{m\sqrt{1+\kappa^2}} + O(h_n^2) + o(h_n^2) \ \ a.s., & \text{if } (x^*, y^*, z^*) \in \widehat{D}_n \cap N_2(x, y, z) \text{ and is more than } h_n^* \text{ away from } L. \end{cases}$$

when n is large enough the first number is negative and the second number is positive almost surely. Define

$$G_1(x, y, z) = \{(x^*, y^*, z^*) \colon (x^*, y^*, z^*) \in \widehat{D}_n \cap O(x, y, z) \text{ and } \vec{\boldsymbol{l}}^T \widehat{\boldsymbol{\beta}}(x^*, y^*, z^*) \le 0\}.$$

$$G_2(x, y, z) = \{(x^*, y^*, z^*) \colon (x^*, y^*, z^*) \in \widehat{D}_n \cap O(x, y, z) \text{ and } \vec{\boldsymbol{l}}^T \widehat{\boldsymbol{\beta}}(x^*, y^*, z^*) > 0\}.$$

Then, when n is large enough, $G_1(x, y, z)$ includes all points that are more than h_n^* away from L and in $\widehat{D}_n \cap N_1(x, y, z)$ and $G_2(x, y, z)$ includes all points that are more than h_n^*

away from L and in $\widehat{D}_n \cap N_2(x,y,z)$. By Theorem 3.1 and the fact that $\frac{m_3}{m} = o(h_n^2)$, the center of $G_1(x,y,z)$ would converge almost surely to some point on P_1 and the center of $G_2(x,y,z)$ would converge almost surely to some point on P_2 and both convergence rates would be $O(h_n)$. *Lemma 4* follows from this result and the results about convergence of $\widehat{\boldsymbol{\beta}}^*$'s in the first paragraph of this proof.

Lemma 5: Besides the conditions in Lemma 4 on f and certain procedure parameters, we further assume that $\widetilde{h}_n = o(1)$, $\frac{h_n}{\widetilde{h}_n} = o(1)$, and $D \cap O(x,y,z)$ is a circular cone, then the local cone fitted by the algorithm described in Section 2.2.3 converges pointwise to $D \cap O(x,y,z)$ almost surely.

Proof of Lemma 5: Without loss of generality, let us assume that the central axis of the true cone is parallel to the x-axis with a vertex at $\boldsymbol{v}$, and the angle between the central axis and any generatrix of the cone is θ. In such cases, the direction of the central axis is $\boldsymbol{\beta}_C = (1,0,0)^T$. From Lemma 2, for a given detected edge pixel (x_l^*, y_l^*, z_l^*) in $O(x,y,z)$, if it is more than $\frac{h_n^*}{\sin\theta}$ away from $\boldsymbol{v}$, then the angle between $\widehat{\boldsymbol{\beta}}_l^*$ and the central axis of the cone is $\theta + O(h_n^*)$ a.s. Therefore, the sample variance, denoted as $\tilde{\sigma}_n^2$, of the inner products of $\{\widehat{\boldsymbol{\beta}}_l^*, l = 1, 2, \ldots, m\}$ and $\boldsymbol{\beta}_C$ is $O((h_n^*)^2)$ a.s. For a given direction $\tilde{\boldsymbol{\beta}}$, if $h_n^*/(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}_C) = o(1)$, then $\tilde{\sigma}_n^2$ would have the property that $(h_n^*)^2/\tilde{\sigma}_n^2 = o(1)$, which is uniformly true for all such $\tilde{\boldsymbol{\beta}}$. So, the direction minimizing $\tilde{\sigma}_n^2$ among all possible directions is $(1, O(h_n^*), O(h_n^*))$ a.s. Therefore, the estimated direction of the central axis of the true cone, as described in item (i) of the algorithm in Section 2.2.3, has the property that $\widehat{\theta} = \theta + O(h_n^*)$ a.s.

From item (iii) of the algorithm in Section 2.2.3, plane $\widetilde{P}$ divides $\widetilde{O}(x,y,z)$ into two parts. Let us define $\widetilde{O}_1(x,y,z)$ to be the part where the vertex $\boldsymbol{v}$ of the cone lies, and the other part is denoted as $\widetilde{O}_2(x, y, z)$. It is clear that the distance of the center of the detected edge pixels in $\widetilde{O}_1(x, y, z)$ from $\widetilde{P}$ is $O(h_n)$, and the center of the detected edge pixels in $\widetilde{O}_2(x, y, z)$ from $\widetilde{P}$ is of the order $\widetilde{h}_n$. So, the center of $\widehat{D}_n \cap \widetilde{O}_1(x, y, z)$, denoted as (c_x^*, c_y^*, c_z^*) in Section 2.2.3, is within $O(h_n^*)$ from the central axis of the true cone, because by Theorem 3.1 all the detected edge pixels are within $O(h_n^*)$ from the true JLSs.

Suppose, the estimated vertex location is $\widehat{\boldsymbol{v}}$. By the fact that the detected edge pixels are within $O(h_n^*)$ from the true JLSs (cf., Theorem 3.1), the orthogonal distance between the fitted cone and the detected edge pixels in $O(x, y, z)$ is $O(\|\widehat{\boldsymbol{v}} - \boldsymbol{v}\|) + O(h_n^*)$ a.s. Moreover, by the algorithm in Section 2.2.3, $\widehat{\boldsymbol{v}}$ is chosen by minimizing the orthogonal distance. By similar arguments to those in the first paragraph of the proof, we have $\|\widehat{\boldsymbol{v}} - \boldsymbol{v}\| = O(h_n^*)$ a.s. By this result and the results obtained in the previous paragraphs, the fitted cone converges pointwise to $D \cap O(x, y, z)$ almost surely.

Proof of Theorem 3.2 From the first part of *Lemma 2*, it is obvious that $\|\widehat{f} - f\|_{\Omega_{\bar{J}, h_n}} = O(h_n^2)$, a.s., under the conditions stated in the theorem. Now, let us consider a given point $(x, y, z) \in J_{h_n} \cap S_\epsilon$ in the following three cases.

Case I: The JLS has unique tangent plane at any of its points in $O(x, y, z)$.

Assume that (x_*, y_*, z_*) is the nearest point on the JLSs to (x, y, z). From the proof of *Lemma 3*, it can be seen that the local plane fitted by the algorithm in Section 2.2.1 converges almost surely to the tangent plane of the JLSs at (x_*, y_*, z_*). By *Lemma 4*, the two half-planes fitted by the algorithm in Section 2.2.2 also converges almost surely to the tangent plane at (x_*, y_*, z_*). So does the fitted cone by the algorithm in Section 2.2.3, as justified by *Lemma 5*.

Recall that $O_1(x,y,z)$ and $O_2(x,y,z)$ are the two parts of $O(x,y,z)$ separated by the true JLS, with $O_1(x,y,z)$ containing the point (x,y,z). Similarly, let us define $E_1(x,y,z)$ to be the part of $O(x,y,z)$ separated by the estimated JLS that contains the point (x,y,z). From the first paragraph of the proof of *Lemma 3*, we know that the number of design points in $E_1(x,y,z)\bigcap O_1(x,y,z)$ and $E_1(x,y,z)\bigcap O_2(x,y,z)$ are of orders $O(n^3h_n^3)$ and $O(n^3h_n^5)$, a.s., respectively. By expression (3), we have

$$\begin{aligned}
\widehat{f}(x,y,z) &= \frac{\sum\limits_{(x_i,y_j,z_k)\in O_1(x,y,z)\cap E_1(x,y,z)} w^*(x_i,y_j,z_k)f(x_i,y_j,z_k)}{\sum\limits_{(x_i,y_j,z_k)\in O_1(x,y,z)\cap E_1(x,y,z)} w^*(x_i,y_j,z_k)}\cdot\frac{|O_1(x,y,z)\cap E_1(x,y,z)|}{|E_1(x,y,z)|} \\
&\quad + \frac{\sum\limits_{(x_i,y_j,z_k)\in O_2(x,y,z)\cap E_1(x,y,z)} w^*(x_i,y_j,z_k)f(x_i,y_j,z_k)}{\sum\limits_{(x_i,y_j,z_k)\in O_2(x,y,z)\cap E_1(x,y,z)} w^*(x_i,y_j,z_k)}\cdot\frac{|O_2(x,y,z)\cap E_1(x,y,z)|}{|E_1(x,y,z)|} \\
&= \left(f(x,y,z)+O(h_n)\right).\left(1+O(h_n^2)\right)+O(h_n^2)\ \ a.s. \qquad (15) \\
&= f(x,y,z)+O(h_n)\ \ a.s.,
\end{aligned}$$

where $w^*(x_i,y_j,z_k)$ denote the weights in the LLK estimator defined in (3), and $|A|$ denotes the number of design points in the region A.

Case II: The JLS has two different one-sided tangent planes at some of its point (x_*,y_*,z_*) in $O(x,y,z)$.

From *Lemma 4*, the two fitted half-planes by the algorithm in Section 2.2.2 converges almost surely to the two one-sided tangent planes with rate $O(h_n)$. Therefore, $RSS_2(x,y,z)/m = O(h_n^2)$, a.s. However, it is obvious that the fitted plane and cone by the algorithms described in Sections 2.2.1 and 2.2.3 both do not converge to the two one-sided tangent planes; further, $RSS_1(x,y,z)/m$ and $RSS_3(x,y,z)/m$ would converge almost surely to two positive constants. So, when n is large enough, we have $BIC_2(x, y, z) < \min(BIC_1(x, y, z), BIC_3(x, y, z))$, a.s. Consequently, the two fitted half-planes by the algorithm in Section 2.2.2 will be selected for estimating the JLS in $O(x, y, z)$ by the BIC procedure (6). From the proof of *Lemma 4*, we can see that $|O_1(x, y, z) \cap E_1(x, y, z)|/|E_1(x, y, z)| = 1 + O(h_n^2)$ a.s., and $|O_2(x, y, z) \cap E_1(x, y, z)|/|E_1(x, y, z)| = O(h_n^2)$ a.s. Therefore, by similar arguments to (15), we have $\widehat{f}(x, y, z) = f(x, y, z) + O(h_n)$ a.s.

Case III: $D \cap O(x, y, z)$ is a circular cone.

By *Lemma 5* and similar arguments to those in Case II, we can show that the BIC procedure (6) would select the fitted cone by the algorithm in Section 2.2.3 for estimating the JLS in $O(x, y, z)$, and consequently $\widehat{f}(x, y, z) = f(x, y, z) + O(h_n)$ a.s. ■