Supplemental file for the paper titled "Jump Detection In Blurred Regression Surfaces"

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Lemma A.1. *Let $\phi(\cdot,\cdot)$ be any continuous function, $K(\cdot,\cdot)$ be a Lipschitz-1 continuous bivariate density kernel function with support $\{(u,v): u^2+v^2 \le 1\}$, and ε_{ij} be i.i.d. random errors from model (2) with mean 0 and variance σ^2. Then, if the bandwidth h_n used in procedure (3) satisfies the condition that $h_n = o(1)$ and $1/(nh_n) = o(1)$, we have*

$$\frac{1}{nh_n} \sum_{(x_i,y_j)\in O_n(x,y)} \varepsilon_{ij}\phi\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right) K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right) \xrightarrow{d} N\left(0, \widetilde{\sigma}^2\right), \text{ as } n \to \infty,$$

where $\widetilde{\sigma}^2 = \sigma^2 \int_{u^2+v^2\le 1} \phi^2(u,v)K^2(u,v)\, dudv$ and (x_i,y_j), $O_n(x,y)$ are defined to be the same as those in (3).

Remark A direct conclusion of Lemma A.1 is that

$$\frac{1}{n^2h_n^2} \sum_{(x_i,y_j)\in O_n(x,y)} \varepsilon_{ij}\phi\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right) K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right) = o\left(\frac{\log(n)}{nh_n}\right) \ a.s.$$

Proof This is a simple application of Lindeberg-Feller conditions. In fact, the terms in the summation are all independent and have the mean 0. Also, we observe that

$$\begin{aligned}\sum_{(x_i,y_j)\in O_n(x,y)} & \mathrm{E}\{\varepsilon_{ij}^2\}\phi\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)^2 K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)^2 \frac{1}{n^2h_n^2} \\ \to \quad & \sigma^2 \int_{u^2+v^2\le 1} \phi^2(u,v)K^2(u,v)\, dudv, \text{ as } n \to \infty.\end{aligned}$$

Next, for any $\delta > 0$, we have

$$\begin{aligned}\sum_{(x_i,y_j)\in O_n(x,y)} & \phi\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)^2 K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)^2 \frac{1}{n^2h_n^2} \cdot \\ & \mathrm{E}\left\{\varepsilon_{ij}^2 I_{\left\{\frac{1}{nh_n}\left|\phi\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)\varepsilon_{ij}\right|>\delta\right\}}\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \quad \frac{C}{n^2h_n^2}\sum_{(x_i,y_j)\in O_n(x,y)} \mathrm{E}\left\{\varepsilon_{ij}^2 I_{\{|\varepsilon_{ij}|>\frac{\delta n h_n}{C}\}}\right\} \\
&\leq \quad \frac{C}{n^2h_n^2}\frac{h_n^2}{1/n^2}\mathrm{E}\left\{\varepsilon_{11}^2 I_{\{|\varepsilon_{11}|>\frac{\delta n h_n}{C}\}}\right\} \\
&\to \quad 0, \qquad \text{as } n\to\infty, \qquad\qquad (A.1)
\end{aligned}
$$

where C is some constant. Thus, all the Lindeberg-Feller conditions are satisfied, and the desired result follows immediately. ∎

Lemma A.2. *Under the condition of Theorem 3.1, the estimated gradient* $(\widehat{b}(x,y),\widehat{c}(x,y))$ *obtained from local linear kernel smoothing procedure (3) has the following properties:*

(i) If (x,y) *is not on any jump location curve, then*

$$
(\widehat{b}(x,y),\widehat{c}(x,y)) \to (f_x'(x,y), f_y'(x,y)), \ \ a.s, \qquad as\ n\to\infty. \tag{A.2}
$$

(ii) If (x,y) *is a nonsingular point on a jump location curve and the jump location curve has a unique tangent line at* (x,y)*, then*

$$
\frac{(\widehat{b}(x,y),\widehat{c}(x,y))}{\sqrt{\widehat{b}(x,y)^2+\widehat{c}(x,y)^2}} \to (-\sin\theta,\cos\theta), \ \ a.s, \ , \qquad as\ n\to\infty, \tag{A.3}
$$

where θ *is the angle formed by the tangent line of the JLC at* (x,y) *and the x-axis.*

(iii) If (x,y) *is a nonsingular point on a jump location curve and the jump location curve has two one-sided tangent lines at* (x,y)*, then*

$$
\frac{(\widehat{b}(x,y),\widehat{c}(x,y))}{\sqrt{\widehat{b}(x,y)^2+\widehat{c}(x,y)^2}} \to \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right),\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right), \ \ a.s, \ , \qquad as\ n\to\infty, \tag{A.4}
$$

where θ_1 *and* θ_2 *are angles formed by the two one-sided tangent lines and the x-axis respectively.*

Proof First, it is not difficult to verify that the solution of procedure (3) has the expressions

$$
\widehat{b}(x,y) = \frac{1}{r_{20}}\sum_{(x_i,y_j)\in O_n(x,y)} (x_i-x)Z_{ij}K\left(\frac{x_i-x}{h_n},\frac{y_j-y}{h_n}\right), \tag{A.5}
$$

$$
\widehat{c}(x,y) = \frac{1}{r_{02}}\sum_{(x_i,y_j)\in O_n(x,y)} (y_j-y)Z_{ij}K\left(\frac{x_i-x}{h_n},\frac{y_j-y}{h_n}\right), \tag{A.6}
$$

where $r_{s_1 s_2} = \sum_{(x_i,y_j)\in O_n(x,y)} (x_i - x)^{s_1}(y_j - y)^{s_2} K\left(\frac{x_i-x}{h_n}, \frac{y_j-y}{h_n}\right)$, for $s_1, s_2 = 0, 1, 2$.

To prove result (A.2), we notice that, for a given point (x, y), if (x, y) is not on any jump location curve, then

$$\mathrm{E}(\widehat{b}(x,y)) = \frac{1}{r_{20}} \sum_{(x_i,y_j)\in O_n(x,y)} P\{f\}(x_i,y_j)(x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right), \quad \text{(A.7)}$$

where

$$\begin{aligned} P\{f\}(x_i,y_j) &= \int\int_{u^2+v^2\le\rho_n^2} p(u,v;x_i,y_j) f(x_i-u, y_j-v)\, dudv \\ &= \int\int_{u^2+v^2\le\rho_n^2} p(u,v;x_i,y_j)\left[f(x_i,y_j) - f'_x(x_i,y_j)u \right. \\ &\quad \left. - f'_y(x_i,y_j)v + O(\rho_n^2)\right] dudv \\ &= f(x_i,y_j) + O(\rho_n^2). \end{aligned} \quad \text{(A.8)}$$

In the last equation of (A.8), we have used the symmetry of p. By (A.7) and (A.8), we have

$$\begin{aligned} &\mathrm{E}(\widehat{b}(x,y)) \\ = &\frac{1}{r_{20}} \sum_{(x_i,y_j)\in O_n(x,y)} [f(x_i,y_j) + O(\rho_n^2)](x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\ = &\frac{1}{r_{20}} \sum_{(x_i,y_j)\in O_n(x,y)} \left[f(x,y) + f'_x(x,y)(x_i - x) + f'_y(x,y)(y_j - y) + \frac{1}{2}f''_{xx}(x,y)(x_i - x)^2 + \right. \\ &\left. f''_{xy}(x,y)(x_i-x)(y_j-y) + \frac{1}{2}f''_{yy}(x,y)(y_j-y)^2 + O(h_n^3)\right](x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\ &+\frac{1}{r_{20}} \sum_{(x_i,y_j)\in O_n(x,y)} O(\rho_n^2)(x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\ = &f'_x(x,y) + O(\rho_n^2/h_n) + O(h_n^2). \end{aligned} \quad \text{(A.9)}$$

In the last equation of the above expression, we have used the results that $r_{s_1,s_2} = 0$, for $s_1, s_2 = 0, 1, 2$ with $s_1 + s_2$ being odd, using the circular symmetry of K, the equal spacing of the design points, and the properties that $r_{20} = O(n^2 h_n^4)$, which can be proved similarly to expression (23) in Proposition 2 of Qiu (2009). Then, by Lemma A.1, we have

$$\frac{1}{n^2 h_n^2} \sum_{(x_i,y_j)\in O_n(x,y)} \varepsilon_{ij}\phi\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = o\left(\frac{\log n}{n h_n}\right), \ a.s., \quad \text{(A.10)}$$

where $\phi(u,v)$ is any continuous function defined in the region $\{(u,v) : u^2 + v^2 \leq 1\}$. By (A.7) and the fact that $r_{20} = O(n^2 h_n^4)$, we have

$$\widehat{b}(x,y) - \mathrm{E}(\widehat{b}(x,y)) = \frac{1}{r_{20}} \sum_{(x_i,y_j)\in O_n(x,y)} \varepsilon_{ij}(x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = o\left(\frac{\log(n)}{nh_n^2}\right), \ a.s. \tag{A.11}$$

Similarly,

$$\widehat{c}(x,y) - \mathrm{E}(\widehat{c}(x,y)) = \frac{1}{r_{02}} \sum_{(x_i,y_j)\in O_n(x,y)} \varepsilon_{ij}(y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = o\left(\frac{\log(n)}{nh_n^2}\right), \ a.s. \tag{A.12}$$

(A.2) is then obtained, after combining (A.11) and (A.12).

To prove (A.3), assume that (x,y) is a nonsingular point on a jump location curve. Then, $O_n(x,y)$ consists of the following three disjoint parts $O_{n,l}(x,y)$, $O_{n,c}(x,y)$, and $O_{n,r}(x,y)$, where $O_{n,c}(x,y)$ is a band of width $2\rho_n$ containing the jump location curve segment, and $O_{n,l}(x,y)$ and $O_{n,r}(x,y)$ are two neighborhoods on its different sides. Since the jump location curve has a unique tangent line at (x,y), difference between the curve and the tangent line will be negligible. Thus, we may assume that the jump location curve segment is a straight line in $O_n(x,y)$ and it forms an angle, denoted by θ, with the x-axis. Then,

$$\begin{aligned}
& \mathrm{E}(\widehat{b}(x,y)) \\
= \ & \frac{1}{r_{20}} \left(\sum_{O_{n,l}(x,y)} + \sum_{O_{n,c}(x,y)} + \sum_{O_{n,r}(x,y)} \right) P\{f\}(x_i,y_j)(x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\
= \ & \frac{1}{r_{20}} \sum_{O_{n,l}(x,y)} [f(x_i,y_j) + O(\rho_n^2)](x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + \\
& \frac{1}{r_{20}} \sum_{O_{n,c}(x,y)} P\{f\}(x_i,y_j)(x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + \\
& \frac{1}{r_{20}} \sum_{O_{n,r}(x,y)} [f(x_i,y_j) + O(\rho_n^2)](x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\
= \ & \frac{1}{r_{20}} \sum_{O_{n,l}(x,y)} [f_-(x,y) + O(h_n) + O(\rho_n^2)](x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(\frac{\rho_n}{h_n^2}\right) +
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{r_{20}} \sum_{O_{n,r}(x,y)} [f_+(x,y) + O(h_n) + O(\rho_n^2)](x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\
= \ & \frac{1}{r_{20}} f_-(x,y) \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) - \\
& \frac{1}{r_{20}} f_-(x,y) \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(\frac{\rho_n^2}{h_n}\right) - \\
& \frac{1}{r_{20}} f_-(x,y) \sum_{O_{n,c}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + \\
& \frac{1}{r_{20}} f_+(x,y) \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(1\right) + O\left(\frac{\rho_n}{h_n^2}\right) \\
= \ & \frac{f_+(x,y) - f_-(x,y)}{r_{20}} \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(1\right) + O\left(\frac{\rho_n}{h_n^2}\right). \qquad (A.13)
\end{aligned}$$

In the second equation of (A.13), (A.8) is used. In the third equation , we have used the results that $r_{20} = O(n^2 h_n^4)$, $P\{f\}(x_i, y_j)$ are uniformly bounded when $(x_i, y_j) \in O_{n,c}(x,y)$, and the fact that the ratio of the area of $O_{n,c}(x,y)$ to the area of $O_n(x,y)$ is of order $O(\rho_n/h_n)$. In the fourth equation, we have used the results that $\sum_{O_{n,r}(x,y)}(x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = O(n^2 h_n^3)$, $\sum_{O_{n,l}(x,y)}(x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = O(n^2 h_n^3)$, $r_{20} = O(n^2 h_n^4)$. In the last equation, we have used the results that $r_{10} = 0$ and $\frac{1}{r_{20}} \sum_{O_{n,c}(x,y)}(x_i - x)K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) = O\left(\frac{\rho_n}{h_n^2}\right)$. By (A.11), we have

$$\widehat{b}(x,y) = \frac{f_+(x,y) - f_-(x,y)}{r_{20}} \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(1\right) + O\left(\frac{\rho_n}{h_n^2}\right) + o\left(\frac{\log(n)}{n h_n^2}\right), \ a.s. \qquad (A.14)$$

Similarly, we can check that

$$\widehat{c}(x,y) = \frac{f_+(x,y) - f_-(x,y)}{r_{02}} \sum_{O_{n,r}(x,y)} (y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O\left(1\right) + O\left(\frac{\rho_n}{h_n^2}\right) + o\left(\frac{\log(n)}{n h_n^2}\right), \ a.s. \qquad (A.15)$$

Notice the following two facts:

$$\frac{h_n}{r_{20}} \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \to \frac{\int_\theta^{\theta+\pi} \ d\varphi \int_0^1 r^2 \cos\varphi K(r) \ dr}{\int_0^{2\pi} \ d\varphi \int_0^1 r^3 \cos^2\varphi K(r) \ dr} = \frac{-2\int_0^1 r^2 K(r) \ dr}{\pi \int_0^1 r^3 K(r) \ dr} \sin\theta. \qquad (A.16)$$

$$\frac{h_n}{r_{02}} \sum_{O_{n,r}(x,y)} (y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \to \frac{\int_\theta^{\theta+\pi} \ d\varphi \int_0^1 r^2 \sin\varphi K(r) \ dr}{\int_0^{2\pi} \ d\varphi \int_0^1 r^3 \sin^2\varphi K(r) \ dr} = \frac{2\int_0^1 r^2 K(r) \ dr}{\pi \int_0^1 r^3 K(r) \ dr} \cos\theta. \qquad (A.17)$$

Therefore,

$$\begin{aligned}\frac{(\widehat{b}(x,y),\widehat{c}(x,y))}{\sqrt{\widehat{b}(x,y)^2+\widehat{c}(x,y)^2}} &= \frac{(h_n\widehat{b}(x,y),h_n\widehat{c}(x,y))}{\sqrt{h_n^2\widehat{b}(x,y)^2+h_n^2\widehat{c}(x,y)^2}} \\ &\to (-\sin\theta,\cos\theta),\ a.s,\end{aligned}$$

which completes the proof of (A.3).

Next, assume that (x,y) is a nonsingular point on a jump location curve, and there exist two one-sided tangent lines of the jump location curve at (x,y), forming angles θ_1 and θ_2, respectively, with the x-axis. See Figure A.1 for a demonstration. The difference between

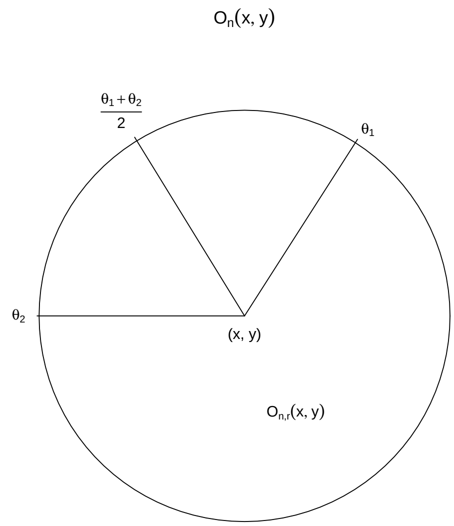

Figure A.1: A demonstration for the case when (x,y) is on a jump location curve that has two one-sided tangent lines at (x,y).

the polygonal line and the jump location curve in $O_n(x,y)$ is negligible when n is sufficiently large. Hence, we may assume that the jump location curve is the same as the polygonal line in $O_n(x,y)$ without loss of generality. By the same arguments in (A.13) and (A.14), we can show that

$$\begin{aligned}\widehat{b}(x,y) &= \frac{f_+(x,y)-f_-(x,y)}{r_{20}}\sum_{O_{n,r}(x,y)}(x_i-x)K\left(\frac{x_i-x}{h_n},\frac{y_j-y}{h_n}\right)+ \\ & O(1)+O\left(\frac{\rho_n}{h_n^2}\right)+o\left(\frac{\log(n)}{nh_n^2}\right),\ a.s. \qquad (A.18)\end{aligned}$$

$$\widehat{c}(x,y) = \frac{f_+(x,y)-f_-(x,y)}{r_{02}} \sum_{O_{n,r}(x,y)} (y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) + O(1) + O\left(\frac{\rho_n}{h_n^2}\right) + o\left(\frac{\log(n)}{nh_n^2}\right), \ a.s. \tag{A.19}$$

Also, we observe the following facts:

$$\begin{aligned}
& \frac{h_n}{r_{20}} \sum_{O_{n,r}(x,y)} (x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\
\to\ & \frac{\int_{\theta_2}^{\theta_1+2\pi} d\varphi \int_0^1 r^2 \cos\varphi K(r)\, dr}{\int_0^{2\pi} d\varphi \int_0^1 r^3 \cos^2\varphi K(r)\, dr} = \frac{\int_0^1 r^2 K(r)\, dr}{\pi \int_0^1 r^3 K(r)\, dr} (\sin\theta_1 - \sin\theta_2) \\
=\ & \frac{2\int_0^1 r^2 K(r)\, dr}{\pi \int_0^1 r^3 K(r)\, dr} \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \cos\left(\frac{\theta_1 + \theta_2}{2}\right).
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
& \frac{h_n}{r_{02}} \sum_{O_{n,r}(x,y)} (y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) \\
\to\ & \frac{\int_{\theta_2}^{\theta_1+2\pi} d\varphi \int_0^1 r^2 \sin\varphi K(r)\, dr}{\int_0^{2\pi} d\varphi \int_0^1 r^3 \sin^2\varphi K(r)\, dr} = \frac{\int_0^1 r^2 K(r)\, dr}{\pi \int_0^1 r^3 K(r)\, dr} (\cos\theta_2 - \cos\theta_1) \\
=\ & \frac{2\int_0^1 r^2 K(r)\, dr}{\pi \int_0^1 r^3 K(r)\, dr} \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \sin\left(\frac{\theta_1 + \theta_2}{2}\right).
\end{aligned} \tag{A.21}$$

Therefore, it follows after combining (A.18)– (A.21) that

$$\begin{aligned}
\frac{(\widehat{b}(x,y), \widehat{c}(x,y))}{\sqrt{\widehat{b}(x,y)^2 + \widehat{c}(x,y)^2}} &= \frac{(h_n\widehat{b}(x,y), h_n\widehat{c}(x,y))}{\sqrt{h_n^2\widehat{b}(x,y)^2 + h_n^2\widehat{c}(x,y)^2}} \\
&\to \left(\cos\left(\frac{\theta_1 + \theta_2}{2}\right), \sin\left(\frac{\theta_1 + \theta_2}{2}\right)\right), \ a.s,
\end{aligned}$$

which finishes the proof of (A.4). ∎

Proof of Theorem 3.1

Let us first prove the theorem for the jump detector LL2K. Let $\widehat{S}_n$ be the set of detected jump points by the jump detector LL2K. For any $(x,y) \in \Omega_{h_n}$, we have

$$\widehat{f}_{LL2K,+}(x,y) = \frac{\sum_{(x_i,y_j)\in U_n(x,y)} \widetilde{w}_{ij}(x,y) Z_{ij}}{\sum_{(x_i,y_j)\in U_n(x,y)} \widetilde{w}_{ij}(x,y)}$$

$$
\begin{aligned}
&= \frac{\sum_{U_n} H\{f\}(x_i, y_j)\widetilde{w}_{ij}(x,y)}{\sum_{U_n} \widetilde{w}_{ij}(x,y)} + \frac{\sum_{U_n} \varepsilon_{ij}\widetilde{w}_{ij}(x,y)}{\sum_{U_n} \widetilde{w}_{ij}(x,y)} \\
&=: I_1(x,y) + I_2(x,y), \qquad \text{(A.22)}
\end{aligned}
$$

where $\sum_{U_n}$ denotes $\sum_{(x_i,y_j)\in U_n}$, U_n is the upper half of $O_n(x,y)$ divided by a line perpendicular to the estimated gradient direction

$$
\widehat{G}(x,y) = \left(\frac{\widehat{c}(x,y)}{\sqrt{\widehat{b}(x,y)^2 + \widehat{c}(x,y)^2}}, \frac{-\widehat{b}(x,y)}{\sqrt{\widehat{b}(x,y)^2 + \widehat{c}(x,y)^2}} \right).
$$

Let $S_{h_n} = \{(x,y) \in \Omega : d_E((x,y), S) \le h_n\}$. Then, for any $(x,y) \in \Omega_{h_n} \setminus S_{h_n}$, $O_n(x,y)$ does not contain any jump point. Let $\widetilde{U}_n(x,y)$ be the half of the $O_n(x,y)$ separated by a line passing (x,y) in the direction perpendicular to the asymptotic direction of $(\widehat{b}(x,y), \widehat{c}(x,y))$, which is discussed in Lemma A.2, and $\widetilde{d}_{ij}$ be the Euclidean distance from (x_i,y_j) to the asymptotic dividing line (thus, $\widetilde{d}_{ij}$ is non-random). For a function ϕ satisfying the condition that $\sup_{u^2+v^2\le 1} |\phi(u,v)| \le b_\phi < \infty$, we have

$$
\begin{aligned}
&\Bigg| \sum_{U_n(x,y)} \phi(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) K(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) L(d_{ij}/h_n) \frac{1}{n^2 h_n^2} - \\
&\sum_{\widetilde{U}_n(x,y)} \phi(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) K(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) L(\widetilde{d}_{ij}/h_n) \frac{1}{n^2 h_n^2} \Bigg| \\
\le\ & \frac{1}{n^2 h_n^2} \Bigg| \sum_{U_n(x,y)} \phi(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) K(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) L(\widetilde{d}_{ij}/h_n) - \\
&\sum_{\widetilde{U}_n(x,y)} \phi(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) K(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}) L(\widetilde{d}_{ij}/h_n) \Bigg| + O\left(\frac{|d_{ij} - \widetilde{d}_{ij}|}{h_n} \right) \\
\le\ & b_\phi \|K\|_\infty \|L\|_\infty \left| \frac{1}{n^2 h_n^2} \sum_{U_n(x,y) \triangle \widetilde{U}_n(x,y)} 1 \right| + O\left(\frac{|d_{ij} - \widetilde{d}_{ij}|}{h_n} \right) \\
=\ & O(\theta_n) = o(1),\ a.s., \qquad \text{(A.23)}
\end{aligned}
$$

where θ_n denotes the acute angle between $(\widehat{b}(x,y), \widehat{c}(x,y))$ and its asymptotic direction and $U_n'(x,y) \triangle \widetilde{U}_n'(x,y) = (U_n'(x,y) \setminus \widetilde{U}_n'(x,y)) \cup (\widetilde{U}_n'(x,y) \setminus U_n'(x,y))$. In the first inequality of

(A.23), we have used the Lipschitz-1 continuity of L. In the last equation, Lemma A.2 has been applied. Now, let

$$\begin{aligned}
\widetilde{b}_{i,j}(x,y) &= [\widetilde{B}_1(x,y)+\widetilde{B}_2(x,y)(x_i-x)+\widetilde{B}_3(x,y)(y_j-y)K\left(\frac{x_i-x}{h_n},\frac{y_j-y}{h_n}\right)L(\widetilde{d}_{ij}/h_n),\\
\widetilde{B}_1(x,y) &= \widetilde{t}_{20}(x,y)\widetilde{t}_{02}(x,y)-\widetilde{t}_{11}(x,y)\widetilde{t}_{11}(x,y),\\
\widetilde{B}_2(x,y) &= \widetilde{t}_{01}(x,y)\widetilde{t}_{11}(x,y)-\widetilde{t}_{10}(x,y)\widetilde{t}_{02}(x,y),\\
\widetilde{B}_3(x,y) &= \widetilde{t}_{10}(x,y)\widetilde{t}_{11}(x,y)-\widetilde{t}_{01}(x,y)\widetilde{t}_{20}(x,y),\\
\widetilde{t}_{s_1,s_2}(x,y) &= \sum_{\widetilde{U}_n(x,y)}(x_i-x)^{s_1}(y_j-y)^{s_2}K\left(\frac{x_i-x}{h_n},\frac{y_j-y}{h_n}\right)L(\widetilde{d}_{ij}/h_n).
\end{aligned}$$

Then, by using similar arguments to those in (A.23), we can check that

$$I_1(x,y)=\frac{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)P\{f\}(x_i,y_j)}{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}+O(\theta_n),\ a.s. \tag{A.24}$$

By using (A.23), we have

$$\begin{aligned}
I_2(x,y) &= \sum_{U_n(x,y)}\frac{\widetilde{w}_{ij}(x,y)\frac{1}{n^4h_n^8}}{\frac{1}{n^6h_n^{10}}\sum_{U_n(x,y)}\widetilde{w}_{ij}(x,y)}\frac{1}{n^2h_n^2}\varepsilon_{ij}\\
&= \sum_{U_n(x,y)}\frac{\frac{1}{n^4h_n^8}\widetilde{b}_{ij}(x,y)+O(\theta_n)}{\frac{1}{n^6h_n^{10}}\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)+O(\theta_n)}\frac{1}{n^2h_n^2}\varepsilon_{ij}\\
&= \sum_{U_n(x,y)}\left(\frac{\frac{1}{n^4h_n^8}\widetilde{b}_{ij}(x,y)}{\frac{1}{n^6h_n^{10}}\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}+O(\theta_n)\right)\frac{1}{n^2h_n^2}\varepsilon_{ij}\\
&= \sum_{U_n(x,y)}\frac{\widetilde{b}_{ij}(x,y)}{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}\varepsilon_{ij}+\frac{1}{n^2h_n^2}\sum_{U_n(x,y)}O(\theta_n)\varepsilon_{ij}\\
&= \sum_{U_n(x,y)}\frac{\widetilde{b}_{ij}(x,y)}{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}\varepsilon_{ij}+O(\theta_n),\ a.s.
\end{aligned} \tag{A.25}$$

In the second equation of (A.25), we have used the results that $\widetilde{B}_1(x,y)=O(n^4h_n^8)$, $\widetilde{B}_2(x,y)=O(n^3h_n^7)$, $\widetilde{B}_3(x,y)=O(n^3h_n^7)$ and $\widetilde{t}_{s_1,s_2}(x,y)=O\left(n^2h_n^{s_1+s_2+2}\right)$ for $s_1,s_2=0,1$. Similar arguments to those in Lemma A.1 can be applied to $\sum_{U_n(x,y)}\frac{\widetilde{b}_{ij}(x,y)}{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}\varepsilon_{ij}$, since $\widetilde{b}_{ij}(x,y)$ is deterministic. Consequently, we have

$$\sum_{U_n(x,y)}\frac{\widetilde{b}_{ij}(x,y)}{\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)}\varepsilon_{ij}\overset{asy.}{\sim}N\left(0,\sum_{U_n(x,y)}\frac{\widetilde{b}_{ij}^2(x,y)}{\left[\sum_{\widetilde{U}_n(x,y)}\widetilde{b}_{ij}(x,y)\right]^2}\right). \tag{A.26}$$

By (A.8), we have that

$$
\begin{aligned}
& \frac{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y) P\{f\}(x_i, y_j)}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} \\
= & \frac{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)(f(x_i,y_j) + O(\rho_n^2))}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} \\
= & \frac{\widetilde{B}_1(x,y)}{|\widetilde{\triangle}|} \sum_{\widetilde{U}_n(x,y)} (f(x,y) + f'_x(x,y)(x_i - x) + f'_y(x,y)(y_j - y) + O(h_n^2) + \\
& O(\rho_n^2)) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) L(\widetilde{d}_{ij}/h_n) + \\
& \frac{\widetilde{B}_2(x,y)}{|\widetilde{\triangle}|} \sum_{\widetilde{U}_n(x,y)} (f(x,y) + f'_x(x,y)(x_i - x) + f'_y(x,y)(y_j - y) + O(h_n^2) + \\
& O(\rho_n^2))(x_i - x) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) L(\widetilde{d}_{ij}/h_n) + \\
& \frac{\widetilde{B}_3(x,y)}{|\widetilde{\triangle}|} \sum_{\widetilde{U}_n(x,y)} (f(x,y) + f'_x(x,y)(x_i - x) + f'_y(x,y)(y_j - y) + O(h_n^2) + \\
& O(\rho_n^2))(y_j - y) K\left(\frac{x_i - x}{h_n}, \frac{y_j - y}{h_n}\right) L(\widetilde{d}_{ij}/h_n) \\
= & f(x,y) + \frac{f'_x(x,y)}{|\widetilde{\triangle}|}(\widetilde{B}_1(x,y)\widetilde{t}_{10}(x,y) + \widetilde{B}_2(x,y)\widetilde{t}_{20}(x,y) + \widetilde{B}_3(x,y)\widetilde{t}_{11}(x,y)) + \\
& \frac{f'_y(x,y)}{|\widetilde{\triangle}|}(\widetilde{B}_1(x,y)\widetilde{t}_{01}(x,y) + \widetilde{B}_2(x,y)\widetilde{t}_{11}(x,y) + \widetilde{B}_3(x,y)\widetilde{t}_{02}(x,y)) + O(h_n^2) + O(\rho_n^2) \\
= & f(x,y) + O(h_n^2) + O(\rho_n^2), \qquad \text{(A.27)}
\end{aligned}
$$

where $|\widetilde{\triangle}| = \widetilde{t}_{00}(x,y)\widetilde{t}_{20}(x,y)\widetilde{t}_{02}(x,y) + \widetilde{t}_{10}(x,y)\widetilde{t}_{01}(x,y)\widetilde{t}_{11}(x,y) + \widetilde{t}_{10}(x,y)\widetilde{t}_{01}(x,y)\widetilde{t}_{11}(x,y) - \widetilde{t}_{01}(x,y)^2\widetilde{t}_{20}(x,y) - \widetilde{t}_{11}(x,y)^2\widetilde{t}_{00}(x,y) - \widetilde{t}_{10}(x,y)^2\widetilde{t}_{02}(x,y)$. In the second equation of (A.27), we have used (A.8). In the last equation, we have used the results that $|\widetilde{\triangle}| = \widetilde{B}_1(x,y)\widetilde{t}_{00}(x,y) + \widetilde{B}_2(x,y)\widetilde{t}_{10}(x,y) + \widetilde{B}_3(x,y)\widetilde{t}_{01}(x,y)$, $\widetilde{t}_{11}(x,y) = 0$ by the symmetry of K and L, $\widetilde{B}_1(x,y)\widetilde{t}_{10}(x,y) + \widetilde{B}_2(x,y)\widetilde{t}_{20}(x,y) + \widetilde{B}_3(x,y)\widetilde{t}_{11}(x,y) = 0$, $\widetilde{B}_1(x,y)\widetilde{t}_{01}(x,y) + \widetilde{B}_2(x,y)\widetilde{t}_{11}(x,y) + \widetilde{B}_3(x,y)\widetilde{t}_{02}(x,y) = 0$, and that $\widetilde{B}_1(x,y) = O(n^4h_n^8)$, $\widetilde{B}_2(x,y) = O(n^4h_n^7)$, $\widetilde{B}_3(x,y) = O(n^4h_n^7)$, $|\widetilde{\triangle}| = O(n^6h_n^{10})$, $\widetilde{t}_{s_1,s_2}(x,y) = O\left(n^2h_n^{s_1+s_2+2}\right)$, for $s_1, s_2 = 0, 1$. All these results can be proved similarly to the result (23) in Proposition 2 in Qiu (2009). Now, after combining (A.24), (A.25), (A.26) and

(A.27), we have the following result:

$$\widehat{f}_{LL2K,+}(x,y) = f(x,y) + O(h_n^2) + O(\rho_n^2) + O(\theta_n) + \xi_n, \tag{A.28}$$

where $\xi_n \overset{asy.}{\sim} N\left(0, \frac{\sum_{U_n(x,y)} \widetilde{b}_{ij}^2(x,y)}{\left[\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)\right]^2}\right)$. Similarly, we have

$$\widehat{f}_{LL2K,-}(x,y) = f(x,y) + O(h_n^2) + O(\rho_n^2) + O(\theta_n) + \eta_n, \tag{A.29}$$

where $\eta_n \overset{asy.}{\sim} N\left(0, \frac{\sum_{V_n(x,y)} \widetilde{b'}_{ij}^2(x,y)}{\left[\sum_{\widetilde{V}_n(x,y)} \widetilde{b'}_{ij}(x,y)\right]^2}\right)$, $\widetilde{b'}_{ij}(x,y)$ is defined similarly to $\widetilde{b}_{ij}(x,y)$. From the proof of Lemma A.2, we know that, if (x,y) is not a jump point, then

$$\theta_n = O(\rho_n^2) + O(h_n^2) + o\left(\frac{\log(n)}{nh_n^2}\right). \tag{A.30}$$

Therefore, for any design point $(x,y) \in \Omega_{h_n} \setminus S_{h_n}$, by (A.28), (A.29) and (A.30), we have

$$\begin{aligned} \widehat{f}_{\text{LL2K},+} - \widehat{f}_{\text{LL2K},-} &= O(h_n^2) + O(\rho_n^2) + o\left(\frac{\log(n)}{nh_n^2}\right) \\ &+ \gamma_n \cdot \sqrt{\frac{\sum_{U_n(x,y)} \widetilde{b}_{ij}^2(x,y)}{\left[\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)\right]^2} + \frac{\sum_{V_n(x,y)} \widetilde{b'}_{ij}^2(x,y)}{\left[\sum_{\widetilde{V}_n(x,y)} \widetilde{b'}_{ij}(x,y)\right]^2}}, \end{aligned} \tag{A.31}$$

where $\gamma_n \overset{asy.}{\sim} N(0,1)$. Also, by using similar arguments to those in (A.23) and the fact that $\widetilde{b}_{ij}(x,y) = O(n^4 h_n^8)$, we have

$$\begin{aligned} \frac{\sum_{U_n(x,y)} w_{ij}^2(x,y)}{[\sum_{U_n(x,y)} w_{ij}(x,y)]^2} &= \frac{n^{10}h_n^{18} \frac{1}{n^{10}h_n^{18}} \sum_{U_n(x,y)} w_{ij}^2(x,y)}{\left[n^6 h_n^{10} \frac{1}{n^6 h_n^{10}} \sum_{U_n(x,y)} w_{ij}(x,y)\right]^2} \\ &= \frac{n^{10}h_n^{18} \left(\frac{1}{n^{10}h_n^{18}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}^2(x,y) + O(\theta_n)\right)}{n^{12}h_n^{20} \left[\left(\frac{1}{n^6 h_n^{10}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y) + O(\theta_n)\right)\right]^2} \\ &= \frac{1}{n^2 h_n^2} \frac{\frac{1}{n^{10}h_n^{18}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}^2(x,y) + O(\theta_n)}{\left[\frac{1}{n^6 h_n^{10}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y) + O(\theta_n)\right]^2} \\ &= \frac{1}{n^2 h_n^2} \left\{ \frac{\frac{1}{n^{10}h_n^{18}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}^2(x,y)}{\left[\frac{1}{n^6 h_n^{10}} \sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)\right]^2} + O(\theta_n) \right\}, \ a.s. \end{aligned} \tag{A.32}$$

Then, it follows that

$$\sqrt{\frac{\sum_{U_n(x,y)} w_{ij}^2(x,y)}{[\sum_{U_n(x,y)} w_{ij}(x,y)]^2} + \frac{\sum_{V_n(x,y)} w_{ij}'^2(x,y)}{[\sum_{V_n(x,y)} w_{ij}'(x,y)]^2}} = O\left(\frac{1}{nh_n}\right), \ a.s. \tag{A.33}$$

By (A.24), (A.25), (A.27), (A.30) and (A.33), we have

$$\begin{aligned} \frac{\text{LL2K}_n(x,y)}{Z_{1-\alpha_n}} &= O\left(\frac{nh_n\rho_n^2}{Z_{1-\alpha_n}}\right) + O\left(\frac{nh_n^3}{Z_{1-\alpha_n}}\right) + o\left(\frac{\log(n)}{h_n Z_{1-\alpha_n}}\right) + o\left(\frac{\log(n)}{Z_{1-\alpha_n}}\right) \\ &= O\left(\frac{nh_n^3}{Z_{1-\alpha_n}}\right), \ a.s., \end{aligned} \tag{A.34}$$

where we have used the conditions that $\frac{\rho_n}{h_n} = o(1)$, and $\frac{\log(n)}{nh_n^4} = o(1)$. Hence, if $\frac{nh_n^3}{Z_{1-\alpha_n}} = o(1)$, any point $(x,y) \in \Omega \setminus S_{h_n}$ will not be flagged as a jump candidate when n is sufficient large.

Now, let us consider a nonsingular design point (x,y) on a jump location curve that has a unique tangent line at (x,y). As discussed in Lemma A.2, we may assume the jump location curve is the same as the tangent line in a small neighbourhood. Let S_{ρ_n} be a band of width $2\rho_n$ that containes S. Then we have in (A.27) and (A.13)

$$\begin{aligned} & \frac{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y) P\{f\}(x_i,y_j)}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} \\ = \ & \frac{\sum_{\widetilde{U}_n(x,y)\setminus S_{\rho_n}} \widetilde{b}_{ij}(x,y) P\{f\}(x_i,y_j)}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} + \frac{\sum_{\widetilde{U}_n(x,y)\bigcap S_{\rho_n}} \widetilde{b}_{ij}(x,y) P\{f\}(x_i,y_j)}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} \\ = \ & \frac{\sum_{\widetilde{U}_n(x,y)\setminus S_{\rho_n}} \widetilde{b}_{ij}(x,y)(f_+(x,y) + O(h_n) + O(\rho_n^2))}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} + O\left(\frac{\rho_n}{h_n}\right) \\ = \ & [f_+(x,y) + O(h_n) + O(\rho_n^2))]\left(1 - O\left(\frac{\rho_n}{h_n}\right)\right) + O\left(\frac{\rho_n}{h_n}\right) \\ = \ & f_+(x,y) + O(h_n) + O(\rho_n/h_n), \end{aligned}$$

where $f_+(x,y)$ denotes the limit of $f(u,v)$ as (u,v) approaching to (x,y) form $\widetilde{U}_n(x,y)$. In the second equation we have used the fact that the ratio of the area of $\widetilde{U}_n(x,y) \bigcap S_{\rho_n}$ to the area of $\widetilde{U}_n(x,y)$ is of order $\frac{\rho_n}{h_n}$. In the third equation, (A.8) has been used. So, we have

$$I_1(x,y) = f_+(x,y) + O(h_n^2) + O(\rho_n/h_n) + O(\theta_n). \tag{A.35}$$

By (A.25) and Lemma A.1, we have

$$I_2(x,y) = \sum_{U_n(x,y)} \frac{\widetilde{b}_{ij}(x,y)}{\sum_{\widetilde{U}_n(x,y)} \widetilde{b}_{ij}(x,y)} \varepsilon_{ij} + O(\theta_n) = o\left(\frac{\log(n)}{nh_n}\right) + O(\theta_n) \text{ a.s.} \tag{A.36}$$

From the proof of Lemma A.2, we know that, when (x, y) is a nonsingular jump point,

$$\theta_n = O(h_n) + O(\rho_n/h_n) + o\left(\frac{\log(n)}{nh_n}\right).$$

Thus,

$$\widehat{f}_{\text{LL2K},+}(x,y) = f_+(x,y) + O(h_n) + O(\rho_n/h_n) + o\left(\frac{\log(n)}{nh_n}\right). \tag{A.37}$$

Similarly, we can derive the result that

$$\widehat{f}_{\text{LL2K},-}(x,y) = f_-(x,y) + O(h_n) + O(\rho_n/h_n) + o\left(\frac{\log(n)}{nh_n}\right), \tag{A.38}$$

where $f_-(x, y)$ is defined similarly to $f_+(x, y)$. Then, a direct conclusion from (A.37), (A.38) and (A.33) is that

$$\begin{aligned} \frac{\text{LL2K}_n(x,y)}{Z_{1-\alpha_n}} &= O\left(\frac{nh_n(f_+(x,y)-f_-(x,y))}{Z_{1-\alpha_n}}\right) \\ &+ O(n\rho_n/Z_{1-\alpha_n}) + O\left(\frac{nh_n^2}{Z_{1-\alpha_n}}\right) + O\left(\frac{\log(n)}{h_n Z_{1-\alpha_n}}\right) \\ &= O\left(\frac{nh_n(f_+(x,y)-f_-(x,y))}{Z_{1-\alpha_n}}\right), \ a.s., \end{aligned} \tag{A.39}$$

where we have used the results that $h_n = o(1)$, $\frac{\rho_n}{h_n} = o(1)$, and $\frac{\log(n)}{nh_n^2} = o(1)$. Thus, in the case when (x, y) is a nonsingular jump point and the jump location curve has a unique tangent line at (x, y), LL2K would detect (x, y) successfully when n is sufficiently large. The parallel result to (A.39) can be derived for the case when the jump location curve has two one-sided tangent lines at (x, y). Therefore, the LL2K jump detector can detect all points in $S \bigcap \Omega_{h_n} \bigcap \overline{J}_{S,h_n}$. And, all points whose Euclidean distances to S are larger than h_n would not be detected. So, when n is large enough, $S \bigcap \Omega_{h_n} \bigcap \overline{J}_{S,h_n}$ is included in $\widehat{S}_n$, and $\widehat{S}_n$ is included in the band of S with width h_n. By similar arguments, it can be shown that this result also holds for the jump detectors LCK, LC2K and LLK. Thus, all results in the theorem are valid. ∎