Some Perspectives On Nonparametric Statistical Process Control

Peihua Qiu
Department of Biostatistics, University of Florida
2004 Mowry Road, Gainesville, FL 32610

Abstract

Statistical process control (SPC) charts play a central role in quality control and management. Many conventional SPC charts are designed under the assumption that the related process distribution is normal. In practice, the normality assumption is often invalid. In such cases, some papers show that certain conventional SPC charts are robust and they can still be used as long as their parameters are properly chosen. Some other papers argue that results from such conventional SPC charts would not be reliable and nonparametric SPC charts should be considered instead. In recent years, many nonparametric SPC charts have been proposed. Most of them are based on the ranking information in process observations collected at different time points. Some of them are based on data categorization and categorical data analysis. In this paper, we give some perspectives on issues related to the robustness of the conventional SPC charts and to the strengths and limitations of various nonparametric SPC charts.

Key Words: Data categorization; Distribution-free; Log-linear modeling; Multivariate distribution; Non-Gaussian data; Nonparametric procedures; Normality; Ordering; Ranking.

1 Introduction

In our daily life, we are often concerned about the quality of products because it is related directly to the quality of our life. In quality control and management, statistical process control (SPC) plays a central role. SPC charts are mainly for monitoring sequential processes (e.g., production lines, internet traffics, medical systems, social or economic status of a population) to make sure that they work stably and satisfactorily. They have been widely used in manufacturing and other industries. See books, such as Hawkins and Olwell (1998), Montgomery (2012), and Qiu (2014), for related discussions.

Since Walter A. Shewhart proposed the first control chart in 1931, many control charts have been proposed in the past more than eighty years, including different versions of the Shewhart chart, cumulative sum (CUSUM) chart, exponentially weighted moving average (EWMA) chart, and the chart based on change-point detection (CPD). See, for instance, Champ and Woodall (1987), Crosier (1988), Crowder (1989), Hawkins (1991), Hawkins et al. (2003), Lowry et al. (1992), Page (1954), Reynolds and Lou (2010), Roberts (1959), Shewhart (1931), and Tracy et al. (1992). A common feature of the control charts discussed in these and many other papers is that the process distribution is assumed to be normal. As pointed out in Subsection 2.3.1 of Qiu (2014), normal distributions play an important role in statistics, because many continuous numerical variables in practice roughly follow normal distributions and much statistical theory has been developed for normally distributed random variables. An intuitive explanation about the reason why many continuous numerical quality variables in our daily life roughly follow normal distributions can be given by using the central limit theorem (CLT). That is, a quality characteristic of a product is often affected by many different factors, including the quality of raw material, labor, manufacturing facilities, proper operation in the manufacturing process, and so forth. So, by the CLT, its distribution should be roughly normal.

In practice, however, there are many variables whose distributions are substantially different from normal distributions. For instance, economic indices and other non-negative indices are often skewed to the right. The lifetimes of products can often be described reasonably well by Weibull distributions which could be substantially different from normal distributions. In multivariate cases, the normality assumption is especially difficult to satisfy, because this assumption implies that all individual variables follow normal distributions and all subsets of them follow joint normal distributions. *A direct conclusion of the multivariate normality assumption is that the regression relationship between any two subsets of the individual variables must be linear, which is rarely valid in practice.* When the normality assumption is invalid, some people think that the conventional control charts based on the normality assumption can still be used as long as their parameters are properly chosen because the charting statistics of these charts are often weighted sample averages of some process observations, and thus the CLT will guarantee that their distributions are close to normal (cf., Borror et al. 1999, Stoumbos and Sullivan 2002, Testik et al. 2003). Some other people think that nonparametric control charts should be considered in cases when no parametric distributions (including the normal distributions) are appropriate for describing the process distribution (cf., Capizzi 2015, Chakraborti et al. 2001, Hackl and Ledolter 1991, Qiu and Hawkins 2001, Qiu and Li 2011a,b). Between the two different opinions about parametric versus nonparametric control charts, which one is appropriate in which cases? We will provide some personal perspectives on this issue in the current paper. In recent years, many nonparametric control charts have been proposed in the literature. Some recent overviews on this topic can be found in Capizzi (2015), Chakraborti et al. (2001), and Qiu (2014, chapters 8 and 9), and some recent papers can be found in the special issue on nonparametric statistical process control of the journal *Quality and Reliability Engineering International* (Chakraborti et al. 2015). Most existing methods are based on the ranking/ordering information of process observations at different time points. Some of them are based on data categorization and on categorical data analysis. In this paper, we will discuss the pros and cons of various different nonparametric control charts.

The rest part of the paper is organized as follows. In Section 2, the robustness property to the normality assumption of certain representative conventional control charts is discussed, and their IC performance is compared to the IC performance of certain representative nonparametric control charts in various cases when the normality assumption is invalid. In Sections 3 and 4, the pros and cons of various univariate and multivariate nonparametric control charts are discussed, respectively. In Section 5, some concluding remarks are given regarding certain future research topics on nonparametric SPC.

2 Robustness of Parametric Control Charts

In this section, we discuss whether conventional parametric SPC charts can still be used when a parametric distribution specified beforehand is invalid. Our discussion focuses mainly on univariate cases when the specified parametric distribution is a normal distribution for simplicity, although most conclusions made in this section should also be appropriate for other univariate parametric distributions and for multivariate cases as well. For control charts designed for monitoring non-normal parametric distributions, such as binomial, Poisson, Weibull, and log-normal distributions, see Gan (1993), Jiang et al. (2011), Reynolds and Stoumbos (2000), Steiner and MacKay (2000), among many others. Also, we focus on the Phase II SPC problem, and the Phase I SPC problem can be discussed similarly. In Phase II SPC research, usually the parameters involved in the in-control (IC) distribution are assumed known. In practice, however, they often need to be estimated from an IC dataset. The impact of the sample size of the IC dataset on the performance of control charts with estimated IC parameters is out of the scope of this paper. See Qiu (2014, chapters 3–5) for related discussions.

In a process monitoring application, if a parametric distribution is available to describe the process distribution well, then a proper parametric control chart can be constructed and used. For instance, if we know that the IC and out-of-control (OC) process distributions are continuous numeric and their probability density (or mass) functions are f_0 and f_1, respectively, then the charting statistic of the CUSUM control chart for detecting the distributional shift from f_0 to f_1 based on a sequence of independent process observations $\{X_1, X_2, \ldots\}$ is

$$G_n = \max\left(0, G_{n-1} + \log\left(f_1(X_n)/f_0(X_n)\right)\right), \quad \text{for } n \geq 1,$$

where $G_0 = 0$. In the case when f_0 and f_1 are the densities of the familiar $N(\mu_0, \sigma^2)$ and $N(\mu_1, \sigma^2)$ distributions with $\mu_1 > \mu_0$, the above statistic becomes

$$C_n^+ = \max\left(0, C_{n-1}^+ + (X_n - \mu_0)/\sigma - k\right), \quad \text{for } n \geq 1, \tag{1}$$

where $C_0^+ = 0$ and $k = (\mu_1 - \mu_0)/(2\sigma)$. It gives a signal when $C_n^+ > h_C$, where $h_C > 0$ is a control limit. In practice, the OC mean μ_1 is often unknown. So, the value of the allowance constant k needs to be specified beforehand. Then, the control limit h_C can be chosen such that a given IC average run length (ARL) value, denoted as ARL_0, is reached. It has been shown that the CUSUM chart (1) has the optimality property that its OC ARL value, denoted as ARL_1, would be the shortest among all control charts with a given value of ARL_0, if the mean shift size is $\delta = (\mu_1 - \mu_0)/\sigma$ and k is chosen $\delta/2$ (cf., Moustakides 1986, Ritov 1990, Yashchin 1993).

Another popular control chart is the EWMA chart. If we are interested in detecting process mean shift from the IC mean μ_0, as in (1), then the EWMA charting statistic is

$$E_n = \lambda(X_n - \mu_0)/\sigma + (1 - \lambda)E_{n-1}, \quad \text{for } n \geq 1, \tag{2}$$

where $E_0 = 0$, and $\lambda \in (0, 1]$ is a weighting parameter. The chart gives a signal of an upward mean shift if $E_n > h_E$, where $h_E > 0$ is a control limit. In (2), the weighting parameter λ needs to be specified beforehand, and the control limit h_E is chosen such that a given value of ARL_0 is reached.

The charts (1) and (2) are for detecting upward mean shifts only. Their counterparts for detecting downward or arbitrary shifts can be defined similarly. In the case when the process distribution is assumed normal, their control limits can be found from tables in text books, such as Tables 4.1 and 5.1 in Qiu (2014). They can also be computed easily by R packages spc and qcc.

Besides the CUSUM and EWMA charts, the Shewhart and CPD charts are also commonly discussed in the literature. Generally speaking, the CUSUM and EWMA charts are more effective than the Shewhart charts for detecting persistent and relatively small shifts, while the Shewhart charts are effective for detecting transient and relatively large shifts. Regarding the CPD charts, they have the advantages that (i) they can provide a reasonably good estimator of the shift position when they give a signal of shift, and (ii) they do not need much prior information about the IC and OC parameters. However, the computation involved in the CPD charts is relatively extensive. In this paper, we mainly focus on the CUSUM and EWMA charts. But, most results presented in the paper should also apply to the Shewhart and CPD charts.

From the above brief description about the conventional control charts, we can see that their construction (cf., (1)) and design (i.e., selection of parameters such as h_C and h_E) depend heavily on the assumed parametric distributions f_0 and f_1. In practice, however, the parametric distributions f_0 and f_1 are often unavailable for describing the IC and OC process distributions, as discussed in Section 1. Real-data examples, in which the conventional normality assumption is invalid, were provided by several papers, including Hawkins and Deng (2010), Qiu and Li (2011b), and Zou and Tsung (2010). Next, let us investigate the IC performance of the conventional CUSUM and EWMA charts in cases when the assumed normality assumption is invalid. In the CUSUM chart (1), we choose $k = 0.5$ and $h_C = 4.389$. In such cases, its nominal ARL_0 is 500 when the IC process distribution is $N(0, 1)$. Suppose that the true IC process distribution is actually the standardized version with mean 0 and standard deviation 1 of the χ^2_{df} distribution, where df is the degrees of freedom changing from 1 to 20. In such cases, the actual ARL_0 values of the CUSUM chart are shown by the solid curve in Figure 1(a). The dashed curve in the plot shows the actual ARL_0 values of the CUSUM chart when the true IC process distribution is the standardized version with mean 0 and standard deviation 1 of the t_{df} distribution, where df is the degrees of freedom changing from 3 to 20. The corresponding results for the EWMA chart (2) with $\lambda = 0.2$ and $h_E = 2.962\sqrt{\lambda/(2-\lambda)}$ are shown in Figure 1(b). From the plots, we can see that the actual ARL_0 values of the two charts could be substantially smaller than the nominal ARL_0 of 500 in most cases considered. These results imply that the related production process would be stopped unnecessarily too often by the two charts. Thus, much time and effort would be wasted in trying to figure out the root causes of the false signals, and the efficiency of the production process would be greatly compromised.

The above example shows that the conventional CUSUM and EWMA charts are inappropriate

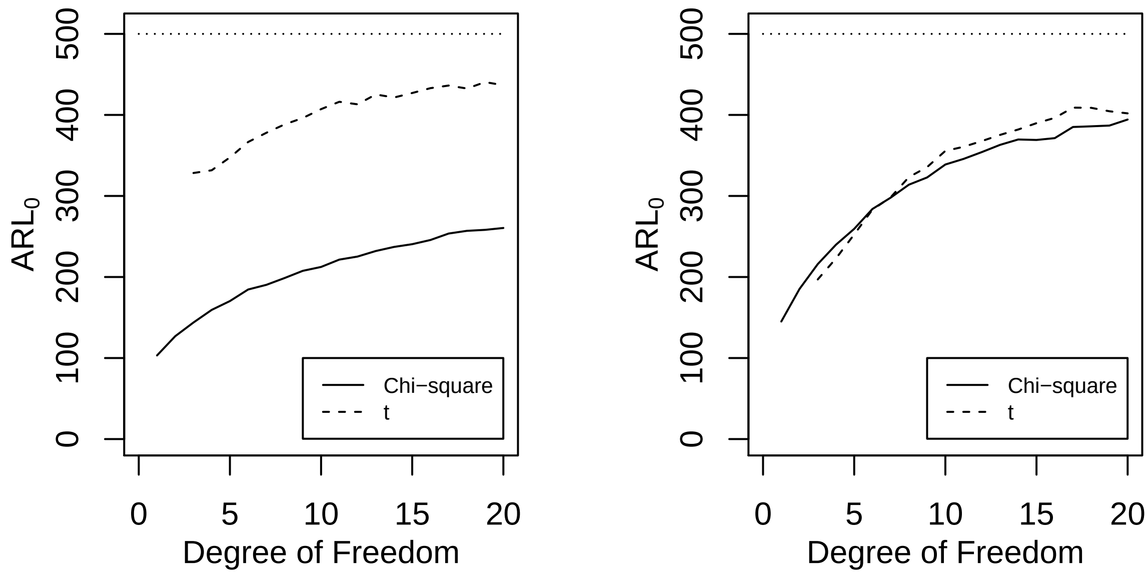

Figure 1: (a) Actual ARL_0 values of the CUSUM chart (1) in cases when the IC process distribution is assumed to be $N(0,1)$ but it is actually the standardized version with mean 0 and standard deviation 1 of the χ^2_{df} distribution (solid curve) or the t_{df} distribution (dashed curve), where df is the degrees of freedom. (b) Corresponding results of the EWMA chart (2). The dotted horizontal line in each plot denotes the nominal ARL_0 value of each chart.

to use in cases when they are designed for normally distributed processes but the actual process distributions are not normal. In such cases, the shift detection power of the charts may be irrelevant because a good power could be due to an overly small ARL_0 value. Also, it may not be realistic to adjust the control limits of the charts so that their actual ARL_0 values are all equal to the given nominal value and then compare the OC performance of the charts, because it is often difficult to know the difference between the actual and nominal ARL_0 values of a chart in practical settings. This is similar to the situation in hypothesis testing, where we should never consider a testing procedure whose actual significance level is larger than the nominal significance level (Lehmann 1986, chapter 3).

In the literature, there have been some discussion about the robustness of the conventional control charts to the normality assumption (e.g., Borror et al. 1999, Humana et al. 2011, Shiau and Hsu 2005, Testik et al. 2003). As discussed in Section 1, some authors think that the conventional control charts should be robust to the normality assumption while some others disagree with them. To address this issue, let us consider the EWMA chart (2). Obviously, its charting statistic can be written as

$$E_n = \lambda \sum_{i=1}^{n} (1-\lambda)^{n-i} \left(\frac{X_i - \mu_0}{\sigma} \right), \quad \text{for } n \geq 1. \tag{3}$$

So, E_n is a weighted average of all process observations up to the current time point n. In Example 5.2 of Qiu (2014), we consider the two-sided version of this chart which gives a signal of process mean shift when $|E_n| > h_E = \rho\sqrt{\lambda/(2-\lambda)}$. For this version of the chart, when (λ, ρ) are chosen to be (0.05,2.216), (0.1, 2.454) or (0.2, 2.635), the ARL_0 values of the chart are 200 in all three cases if the IC process distribution is assumed to be $N(0,1)$. If the true IC process distribution is actually the standardized version with mean 0 and standard deviation 1 of the χ^2_{df} distribution, the actual ARL_0 values of the chart are shown in Figure 2. From the figure, it can be seen that the actual ARL_0 values are quite close to the nominal ARL_0 value of 200 when λ is chosen small (e.g., 0.1 or 0.05) and df is not too small (i.e., the process distribution is not too skewed). In practice, however, it is often hard to know how different the actual process distribution is from a normal distribution. Therefore, it is hard to determine how small λ should be chosen in a given application. Furthermore, if λ is chosen small, the corresponding chart would be ineffective in detecting relatively large shifts, as well demonstrated in the literature (cf., Figure 5.4 in Qiu (2014)).

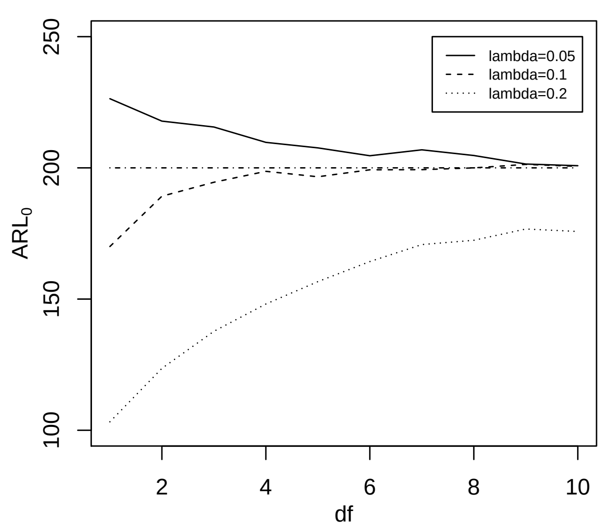

Figure 2: Actual ARL_0 values of the two-sided EWMA chart (2) in cases when the actual IC process distribution is the standardized version with mean 0 and standard deviation 1 of the χ^2_{df} distribution. The dot-dashed horizontal line denotes the nominal ARL_0 value of the chart when the IC process distribution is assumed to be $N(0,1)$.

In cases when the true IC process distribution is unknown but an IC dataset is available, Qiu and Zhang (2015) discussed a possible approach to overcome the difficulty due to the violation of the normality assumption. By that approach, a transformation is first defined based on the IC dataset so that the transformed process observations roughly follow a normal distribution, and then the conventional control charts can be applied to the transformed process observations. Transformations considered were either parametric or nonparametric. The nonparametric transformations were found inappropriate to use because they could be regarded as parametric transformations with infinite number of parameters and their estimators from an IC dataset of a reasonable size would have large variability. The major conclusions about parametric transformations were that they should be used with care, and a parametric transformation defined from the IC data could generally shrink the difference between the actual and the nominal ARL_0 values, especially when the IC data size was large; but, it was difficult to find a parametric transformation that was good for all cases with non-normal data. For a related discussion, see Qiu and Li (2011a).

As a summary of our discussion in this section, in cases when the normality assumption is invalid, conventional control charts designed based on that assumption should be used with care because their IC performance could be substantially different from what we would expect. The transformation approach is helpful in alleviating the problem. But, it cannot solve the problem satisfactorily in many cases.

3 Univariate Nonparametric Control Charts

To overcome the limitations of the parametric control charts discussed in the previous sections, there have been many nonparametric control charts developed in recent years that do not require specification of a parametric form for describing the process distribution. Some of them are even distribution-free in the sense that distributions of their charting statistics do not depend on the true process distribution. In many papers, however, people do not always make a clear distinction between the terminologies of "nonparametric control charts" and "distribution-free control charts." Instead, both terminologies are referred to the charts that can be used when the process distribution does not have a parametric form. Some "nonparametric control charts" may not be distribution-free, in the sense that their design may still depend on the process distribution, although a parametric form for the process distribution is not required. In this section, we discuss some representative univariate nonparametric control charts and their major properties. For simplicity of presentation, our discussion focuses mainly on Phase II SPC, although Phase I SPC is also important and can be discussed similarly. For Phase I univariate nonparametric control charts, see papers such as Capizzi and Masarotto (2013), Graham et al. (2010), Jones-Farmer et al. (2009), and Ning et al. (2015).

Research on nonparametric SPC in univariate cases has a quite long history. For some early papers, see, e.g., Alloway and Raghavachari (1991), Bakir and Reynolds (1979), Hackl and Ledolter (1991), and Park and Reynolds (1987). Many methods in this area are based on the ranking/ordering information within different batches of the observed batch data of a related process. Let the batch of m independent and identically distributed process observations at the current time point n be

$$X_{n1}, X_{n2}, \ldots, X_{nm}, \qquad \text{for } n \geq 1.$$

Then, the sum of the Wilcoxon signed-ranks within the n-th batch of observations is defined to be

$$\psi_n = \sum_{j=1}^{m} \text{sign}(X_{nj} - \eta_0) R_{nj}, \tag{4}$$

where η_0 is the IC median of the process distribution, $\text{sign}(u) =$ -1,0,1, respectively, when $u < 0, = 0, > 0$, and R_{nj} is the rank of $|X_{nj} - \eta_0|$ in the sequence $\{|X_{n1} - \eta_0|, |X_{n2} - \eta_0|, \ldots, |X_{nm} - \eta_0|\}$. Obviously, the absolute value of ψ_n would be small if the process is IC, because the positive and negative values in the summation of (4) will be roughly the same and they will be mostly cancelled out. If there is an upward mean shift before or at time n, then the value of ψ_n will tend to be positively large. Its value will be negatively large if there is a downward mean shift. So, ψ_n in (4) carries useful information about potential mean shifts. Based on this statistic, Bakir (2004) and Chakraborti and Eryilmaz (2007) proposed their Shewhart charts for detecting process mean shifts, Bakir and Reynolds (1979) and Li et al. (2010) suggested two CUSUM charts, and Graham et al. (2011) and Li et al. (2010) discussed the related EWMA charts. In the charts discussed by Li et al. (2010), a reference sample is assumed to be available. When a reference sample is available, Chakraborti et al. (2004) and Mukherjee et al. (2013) proposed the so-called precedence charts by comparing the observations in a given batch of Phase II data with the observations in the reference sample. Because the IC distribution of ψ_n does not depend on the specific form of the IC process distribution as long as the IC process distribution is symmetric, most charts based on ψ_n are distribution-free in that sense. Besides ψ_n, some alternative nonparametric statistics are possible. For instance, Amin et al. (1995) and Lu (2015) considered using the sign test statistic in their nonparametric control charts, Chowdhury et al. (2014) used the Cucconi test statistic, and Liu et al. (2014) used sequential ranks. In cases when there is only one observation at each time point, several nonparametric control charts have been proposed using the ranking information among observations at different time points. For instance, Zou and Tsung (2010) proposed a nonparametric EWMA chart based on a nonparametric likelihood ratio test. Their chart is self-starting and capable of detecting all distributional shifts. Hawkins and Deng (2010) proposed a nonparametric CPD chart based on the nonparametric Mann-Whitney two-sample test. Ross et al. (2011) and Ross and Adams (2012) proposed several nonparametric CPD charts for detecting mean, variance, and other distributional shifts.

All the nonparametric control charts mentioned above are based on the ranking information among different process observations. Another type of nonparametric control charts takes an alternative approach by first categorizing the original observations and then using tools of categorical data analysis for constructing control charts (cf., Qiu 2008, Qiu and Li 2011b). Assume that the process observations at the current time point n are $\mathbf{X}_n = (X_{n1}, X_{n2}, \ldots, X_{nm})'$, where the batch size m can be 1 and the observations are numerical. First, we categorize the original observations into the following c intervals: $(-\infty, \xi_1], (\xi_1, \xi_2], \ldots, (\xi_{c-1}, \infty)$. Let $Y_{njl} = I(X_{nj} \in (\xi_{l-1}, \xi_l])$, for $l = 1, 2, \ldots, c$, where $\xi_0 = -\infty$ and $\xi_c = \infty$. Then, $\mathbf{Y}_{nj} = (Y_{nj1}, Y_{nj2}, \ldots, Y_{njc})'$ has one component equal to 1, the remaining components are all 0, and the index of the component "1" is the index of the interval that X_{nj} belongs to. The vector $\mathbf{Y}_{nj}$ can be regarded as the categorized data of X_{nj}, and its distribution can be described by $\mathbf{f} = (f_1, f_2, \ldots, f_c)'$, where f_l denotes the probability that X_{nj} belongs to the l-th interval, for $l = 1, 2, \ldots, c$. When the process is IC, the distribution of $\mathbf{Y}_{nj}$ is denoted as $\mathbf{f}^{(0)} = (f_1^{(0)}, f_2^{(0)}, \ldots, f_c^{(0)})'$. Then, $g_{nl} = \sum_{j=1}^{m} Y_{njl}$ denotes the observed count of original observations at the current time point that belong to the l-th interval, and $mf_l^{(0)}$ is the corresponding expected count. The Pearson's chi-square statistic $\widetilde{X}_n^2 = \sum_{l=1}^{p} (g_{nl} - mf_l^{(0)})^2 / (mf_l^{(0)})$ measures their discrepancy. It has been justified that a mean shift in the original process distribution will result in a shift in the distribution of $\mathbf{Y}_{nj}$ under some mild conditions. Then, based on $\widetilde{X}_n^2$, a CUSUM chart can be constructed in a similar way to that in Crosier (1988) for detecting process mean shifts, and it was shown that this chart could also detect process variance shifts (cf., Qiu and Li 2011b). As a sidenote, it is obvious that (i) Shewhart and EWMA charts can also be constructed easily based on $\widetilde{X}_n^2$, and (ii) these charts can be used for detecting distributional shifts of processes with categorical observations by simply skipping the data categorization step. From the above description, the IC properties of the charts based on the discretized data $\mathbf{Y}_{nj}$ depend on the $c-1$ parameters $f_1^{(0)}, f_2^{(0)}, \ldots, f_{c-1}^{(0)}$ (note: $f_c^{(0)} = 1 - \sum_{l=1}^{c-1} f_l^{(0)}$). As well discussed in the categorical data analysis (cf., Agresti 2002), the Pearson's chi-square statistic $\widetilde{X}_n^2$ would be most powerful for detecting distributional shifts in $\mathbf{Y}_{nj}$ when we choose $f_l^{(0)} = 1/c$, for all l. So, these parameter values are always recommended. In that sense, the charts based on the discretized data $\mathbf{Y}_{nj}$ are distribution-free. However, in practice, in order to categorize the observations, we still need to determine the cut points $\{\xi_l, l = 1, 2, \ldots, c-1\}$, which are the quantiles of the IC process distribution F_0 (e.g., ξ_1 is the $f_1^{(0)}$-quantile). In some cases, these parameters can be determined easily. For example, if we know that the IC process distribution is symmetric and $c = 2$, then ξ_1 equals the IC process mean or median. But, in a general case, they depend on F_0 and need to be estimated from an IC dataset, which is similar to the case with a conventional CUSUM chart in which the IC mean and variance need to be estimated from an IC dataset. In that sense, the charts based on $\mathbf{Y}_{nj}$ are not distribution-free, although only some IC parameters need to be determined in advance. Qiu and Li (2011b) showed that the performance of the charts based on $\mathbf{Y}_{nj}$ would be stable in cases when $M \geq 200$ and $m = 5$, where M is the sample size of the IC data.

One common feature of most nonparametric control charts is that their charting statistics are discrete. A direct consequence of the discreteness is that these charts may not be able to reach a specific ARL_0 value. For instance, when $m = 5$, the possible values of ψ_n in (4) are $\{-15, -13, \ldots, -1, 1, \ldots, 13, 15\}$. If we consider using the upward CUSUM chart with the charting statistic

$$C_n^+ = \max\left(0, C_{n-1}^+ + \psi_n - k\right), \quad \text{for } n \geq 1, \tag{5}$$

where $C_0^+ = 0$ and $k = 9$, and the chart gives a signal when $C_n^+ > h$, then the possible ARL_0 values of the chart are 38.40, 53.76, 72.98, 110.87, 155.36 and 249.32 when h is chosen in $(0, 12]$. In such cases, the chart cannot reach the ARL_0 value of 200, for instance. In some cases, we may prefer to use a specific ARL_0 value (e.g., 200, 370), especially when we want to compare several control charts. In such cases, Qiu and Li (2011b) proposed a simple modification of a charting statistic so that its discreteness can be reduced dramatically without altering its OC properties. By this idea, the statistic C_n^+ in (5) can be modified in the way described as follows. Let $b_{n1}, b_{n2}, \ldots, b_{nm}$ be i.i.d. random numbers from the distribution $N(0, \nu^2)$, where $\nu > 0$ is a small number (e.g., 0.01). Then, ψ_n in (5) can be replaced by

$$\psi_n^* = \sum_{j=1}^{m} \text{sign}(X_{nj} - \eta_0)(R_{nj} + b_{nj}). \tag{6}$$

From (6), when the process is IC, the mean of ψ_n^* is still 0, but ψ_n^* can take more different values than ψ_n. Thus, it is less discrete than ψ_n. The discreteness of ψ_n^* is controlled by the value of ν in the following sense. When ν is chosen smaller, ψ_n^* can take less different values and thus it is more discrete, and it is less discrete when ν is chosen larger. After the process becomes OC, because ν is chosen small, the OC performance of ψ_n^* is almost the same as that of ψ_n. Again, the difference between the OC performance of ψ_n^* and ψ_n is controlled by ν. When ν is chosen smaller, ψ_n^* and ψ_n will be more similar, and their OC performance is more different if ν is chosen larger. In practice, ν should be chosen as small as possible, as long as a specific ARL_0 value can be achieved within a desired precision. To demonstrate these results, let us consider the following example. Assume that the IC process distribution is the standardized version of the t_3 distribution with mean 0 and variance 1, and we consider using the modified version of the chart (5) with $k = 9$ and with ψ_n replaced by ψ_n^* in (6). If the given ARL_0 value is 200 and ν in the distribution of b_{nj} is chosen to be 0.01, then the actual ARL_0 value is computed to be 199.936 when $h = 7.992$, based on 10,000 replicated simulations. We tried other commonly used ARL_0 values as well, such as 370 and 500, they can all be reached within a good precision too. Now, assume that the process mean changes from 0 to δ, and δ varies from 0 to 2 with a step of 0.2. The ARL_1 values of the chart (5) with and without using the modification are presented in Figure 3 by the solid and dashed lines, respectively. It can be seen that the two sets of ARL_1 values in the two cases are indeed close to each other, and the difference gets smaller when δ increases. One obvious limitation of the modification procedure discussed above is that the ARL_0 value of the modified chart depends on the random numbers b_{nj} and thus it is random. However, as long as we compute ARL_0 based on a large number (e.g., 10,000) of replicated simulations, this randomness should be small.

To use the nonparametric control charts, some information in the original observations is lost and not used in process monitoring. This is true for all nonparametric control charts, including the ones based on data ranking/ordering and the ones based on data categorization. For the nonparametric control charts based on data ranking/ordering, so far we have not found any method to control the amount of lost information. For the ones based on data categorization, the amount of lost information is controlled by the number of intervals, c, used in data categorization. If c is chosen larger, then the amount of lost information will be smaller; the amount of lost information will be larger otherwise. If c is chosen larger, however, more IC parameters (e.g., $\xi_1, \xi_2, \ldots, \xi_{c-1}$) need to be estimated from the IC data. Qiu and Li (2011b) discussed the selection of c. The

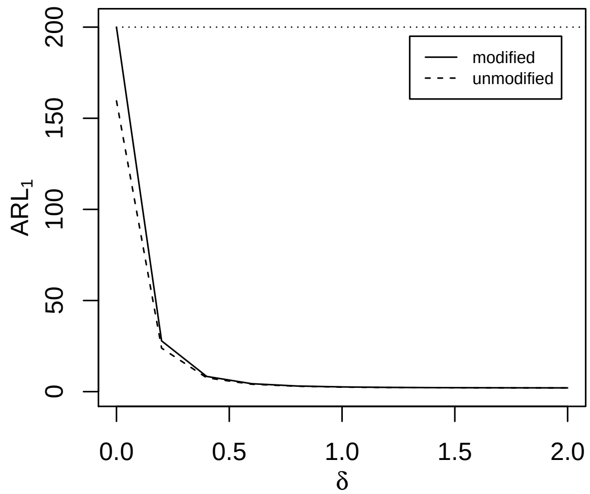

Figure 3: The solid line denotes the ARL_1 values of the modified version of the chart (5) with $k = 9$, $h = 7.992$, and $\nu = 0.01$ ($ARL_0 = 200$ in such cases) in cases when the IC process distribution is the standardized version of the t_3 distribution with mean 0 and variance 1, the process has a mean shift of δ at the initial time point, and δ changes from 0 to 2 with a step of 0.2. The dashed line denoted the ARL_1 values of the unmodified version with $k = 9$ and $h = 7.992$ in the same cases. The dotted line on top denotes the assumed ARL_0 value of 200.

major conclusions are that (i) when $m = 1$ and the process distribution is quite symmetric, c can be chosen as small as 5, (ii) when $m = 1$ and the process distribution is quite skewed, c should be chosen larger, but the performance of the related chart would hardly be improved for choosing $c > 10$, and (iii) when $m > 1$, c can be chosen smaller than its value in cases when $m = 1$.

Next, we compare the numerical performance of the following several representative control charts: i) the nonparametric CUSUM chart based on data categorization with $c = 5$ (denoted as C-CUSUM), (ii) the Lepage-type nonparametric control chart based on change-point detection that was proposed in Ross et al. (2011) (denoted as LP-CPD), (iii) the conventional two-sided CUSUM chart based on the normality assumption (denoted as N-CUSUM), and (iv) the conventional EWMA chart (2) (denoted as R-EWMA). In all charts, ARL_0 is chosen to be 500, and all results are computed from batch data with batch size $m = 5$ based on 10,000 replicated simulations. The charts C-CUSUM and LP-CPD represent two different types of nonparametric control charts that are based on data categorization and data ranking, respectively. The IC parameters in C-CUSUM (i.e., $\{\xi_1, \xi_2, \ldots, \xi_{c-1}\}$) are estimated from an IC data of size $M = 500$. Once its allowance constant (cf., k in (1)) is chosen, its control limit is determined by a bootstrap procedure with a bootstrap sample size 10,000 from the IC data such that ARL_0 reaches 500. The conventional charts N-

CUSUM and R-EWMA are set up in a similar way. Namely, their IC parameters (i.e., μ_0, σ, and η_0) are estimated from the IC data, and their control limits are determined by the bootstrap procedure from the IC data such that $ARL_0 = 500$. The chart LP-CPD can be implemented in the R-package cpm. It does not need to estimate the IC parameters. Instead, it requires a "burn-in" sample. The default "burn-in" sample size is 20. In this example, we consider both the default version and the version with "burn-in" sample size of 120 (i.e., $120 * 5 = 600$ observations), denoted as LP-CPD and LP-CPD1, respectively. Each of the three charts C-CUSUM, N-CUSUM and R-EWMA has a smoothing parameter involved (i.e., the allowance constant k in the two CUSUM charts and the weighting parameter λ in the EWMA chart). If we compare their performance by specifying their parameter values in advance, then the comparison may not be meaningful. For instance, the C-CUSUM chart with $k = 0.1$ may not be comparable with the R-EWMA chart with $\lambda = 0.1$. Even the two CUSUM charts C-CUSUM and N-CUSUM with a same k value are not comparable because the former is based the categorized data while the latter is based on the original data. The CUSUM and EWMA charts are not comparable with the CPD charts LP-CPD and LP-CPD1 either because the former have their smoothing parameters involved while the latter do not have such parameters. To compare all charts fairly, in this paper the related smoothing parameter is chosen such that the ARL_1 value of each chart for detecting a given shift reaches the minimum. Namely, the optimal performance of the charts is compared here, as did in papers such as Qiu (2008). The IC process distribution is chosen to be the standized version with mean 0 and variance 1 of the following 4 distributions: $N(0, 1), t_4, \chi_1^2$ and χ_4^2. The t distribution represents symmetric distributions with heavy tails, while the two chi-square distributions represent skewed distributions with different skewness. Then, for each IC process distribution, we consider 10 mean shifts from -1.0 to 1.0 with step 0.2. The calculated ARL_1 values of the four charts are shown in Figure 4.

From Figure 4, we can have the following conclusions. First, in cases when the normality assumption is valid (plot (a)), the N-CUSUM performs the best, as expected, and the R-EWMA chart also performs reasonably well, especially when the shift sizes are small. The chart LP-CPD is not good, but the version LP-CPD1 is reasonably good, especially when the shift size is large. In this case, the conventional charts N-CUSUM and R-EWMA should be used, the nonparametric charts LP-CPD1 and C-CUSUM will lost some efficiency, but the lost of efficiency is quite small when the shift size is relatively large (e.g., the magnitude of the shift is larger than 0.8). In cases

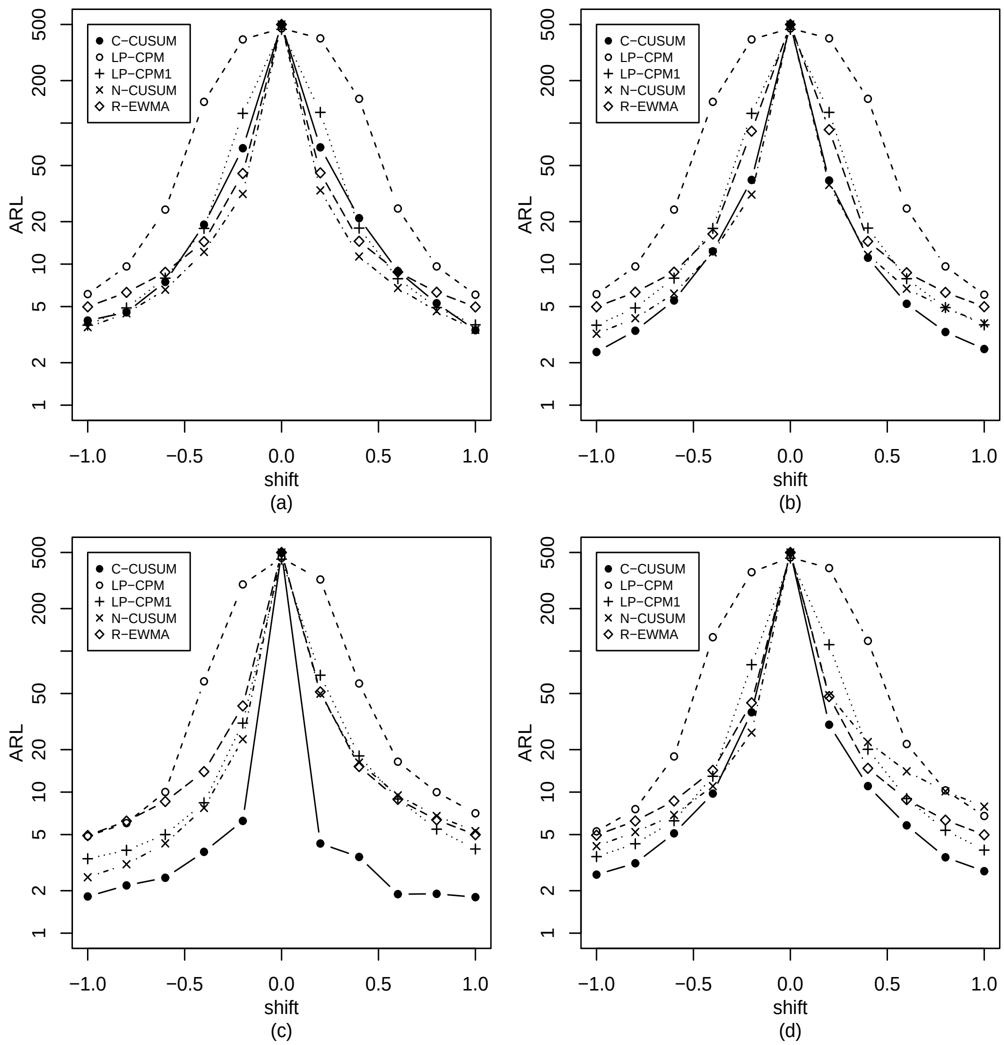

Figure 4: Optimal OC ARL values of five control charts when the IC ARL is 500, $m = 5$, and the actual IC process distribution is the standardized version of $N(0, 1)$ (plot (a)), $t(4)$ (plot (b)), $\chi^2(1)$ (plot (c)), and $\chi^2(4)$ (plot (d)). Scale on the y-axis is in natural logarithm.

when the process distribution is symmetric with heay tails (plot (b)), the chart LP-CPD does not perform well, the version LP-CPD1 is reasonably good when the shift size is large, the conventional chart N-CUSUM performs well only when the shift is small, the other conventional chart R-EWMA is worse than N-CUSUM consistently, and the chart C-CUSUM is the best almost all the times. So, in this case, the conventional chart N-CUSUM is recommended only when the shift is expected to be small, and the chart C-CUSUM should be the one to use. In cases when the process distribution is skewed (plots (c) and (d)), the chart C-CUSUM still performs well in all cases, and the other charts are not always effective. These results are intuitively reasonable for the following reasons. 1) When the normality assumption is valid, the nonparametric charts LP-CPD, LP-CPD1 and C-CUSUM are not as effective as the N-CUSUM chart because the former would lose information by using ranks or data categorization. 2) When the normality assumption is invalid, the convensional charts N-CUSUM and R-EWMA would not be effective because they are constructed based on the normal distribution densities. See the general formula for the CUSUM charting statistic immediately before expression (1). Therefore, when the normality assumption is violated, its performance will not be good.

Because the nonparametric control charts do not use all information in the original data, should they be avoided in practice? To answer this question, let us briefly discuss a similar issue in the context of hypothesis testing. In that context, the null hypothesis is usually protected because it should have been tested extensively in the past. A hypothesis testing procedure is acceptable only when its type-I error probability is smaller than or equal to a prespecified significance level α. In such cases, the smaller its type-II error probabilities, the better. The degree of protection of the null hypothesis is controlled by α. In applications where the consequence of type-I error could be very serious (e.g., applications related to new drugs), α should be chosen small (e.g., 0.01). Otherwise, α could be chosen relatively large. In most applications, people use the default value of α which is 0.05. Now, in the context of Phase II SPC, the IC status of the process at the beginning of online monitoring is confirmed in Phase I analysis. So, it should be protected as well, in the sense that an online monitoring procedure is acceptable only when its actual ARL_0 value is not smaller than the prespecified level. Otherwise, the false signal rate could be higher than what is expected. Of course, it is not good either if the actual ARL_0 value of a chart is much larger than the prespecified level because this could result in more defective products than expected. Such a chart, however, is easier to figure out because its ARL_1 values are usually also large and it is in a disadvantageous status in comparison with competing charts. In cases when the consequence of defective products is very serious, a small prespecified ARL_0 value should be used. Otherwise, the prespecified ARL_0 value can be chosen relatively large. *As shown in Figure 1, when the assumed parametric distribution is invalid, the actual ARL_0 values of the conventional parametric control charts could be substantially smaller than the prespecified ARL_0 value. In such cases, they should be avoided and the nonparametric control charts should be used. The loss of information is just the price to pay for the nonparametric control charts to reach the prespecified ARL_0 level without specifying a parametric form for describing the IC process distribution. One important future research topic is to minimize the lost information while keeping the favorable* *properties of the nonparametric control charts.*

At the end of this section, we would like to mention that the IC distributions of nonparametric charting statistics are often difficult to derive analytically. So, the bootstrap and other numerical approaches are often used for this purpose (e.g., Ambartsoumian and Jeske 2015, Chatterjee and Qiu 2009). Some authors also considered using some conventional charting statistics, such as the CUSUM and EWMA charting statistics (1) and (2) and the Hotelling's statistic in multivariate cases, in cases when the normality assumption is invalid. In such cases, because the conventional IC distributions of these statistics under the normality assumption are no longer valid, various bootstrap algorithms were developed for computing their control limits (e.g., Noorossanaa and Ayoubi 2011, Phaladiganon et al. 2011). Such control charts would have good IC properties, but their OC properties may not be as good as we would expect, because their charting statistics are constructed under the normality assumption or other related distributional assumptions and have good properties only when such assumptions are valid. See some numerical results about N-CUSUM and R-EWMA in Figure 4 and some related discussions in Jones and Woodall (1998).

4 Multivariate Nonparametric Control Charts

Quality is a multifaceted concept. In most SPC applications, we are concerned with multiple quality characteristics. In that sense, *SPC research should focus on multivariate cases.* When process observations are multivariate, it is rare in practice that their distribution is multivariate normal, as explained in Section 1. In the statistical literature, existing methods for describing multivariate non-normal data or transforming multivariate non-normal data to multivariate normal data are limited. If a control chart based on the normality assumption is used in cases when the assumption is invalid, then the actual ARL_0 value of the chart could be substantially different from the assumed one, as demonstrated by Figure 1 in univariate cases. See the related discussion in Section 9.1 of Qiu (2014). Therefore, development of multivariate nonparametric control charts is exceptionally important. In this section, we discuss some existing methods for handling this problem and discuss their major properties. A small comparison study among some of them is also presented. It should be pointed out that multivariate nonparametric SPC is much more challenging than its univariate counterpart, partly because i) the multiple quality characteristics could have a complicated correlation structure among them, ii) potential shifts in the process distribution could have infinitely many possible directions, and so forth. Thus, it requires much future research effort to solve this problem properly in different scenarios and compare different methods in a comprehensive manner.

As in univariate cases, one natural idea to handle the multivariate nonparametric SPC (MNSPC) problem is to use the ranking/ordering information in the process observations. In multivariate cases, there are two types of ranking information. The *longitudinal ranking* refers to the one among observations at different time points, and the *cross-component ranking* refers to the ranking across different components of a multivariate observation at a given time point. The first MNSPC method based on longitudinal ranking might be the one by Liu (1995) using the concept of data depth (cf., Liu et al. 2004, Liu and Singh 1993). One fundamental difference between univariate and multivariate cases is that process observations in univariate cases are naturally ordered on the number line and such ordering is not well defined in multivariate cases. One major purpose of data depth is to establish an order among multivariate observations. There are several different definitions of data depth in the literature. The Mahalanobis depth of a p-dimensional point $\mathbf{y}$ with respect to a p-dimensional distribution F_0 is proportional to the reciprocal of the Mahalanobis distance $(\mathbf{y}-\boldsymbol{\mu}_0)'\Sigma_0(\mathbf{y}-\boldsymbol{\mu}_0)$, where $\boldsymbol{\mu}_0$ and Σ_0 are the mean vector and covariance matrix of F_0. Then, p-dimensional observations can be ordered by their Mahalanobis distances: the smaller the Mahalanobis distance, the closer to the center of the distribution F_0. One obvious limitation of this depth is that it ignores the shape of F_0 completely. To overcome this limitation, Liu (1990) defined the simplicial depth of $\mathbf{y}$ as $P(\mathbf{y} \in \mathcal{S}(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{p+1}))$, where $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{p+1})$ is a simple random sample from F_0, $\mathcal{S}(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{p+1})$ is an open simplex with vertices at $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{p+1}$, and $P(\cdot)$ is the probability under F_0. In practice, F_0 is often unknown. Instead, we may have a reference sample $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_M)$ from F_0. In such cases, Liu (1990) suggested replacing the probability in the definition of simplicial depth by the proportion $[1/\begin{pmatrix} M \\ p+1 \end{pmatrix}] \sum I\left(\mathbf{y} \in \mathcal{S}(\mathbf{Y}_{i_1}, \mathbf{Y}_{i_2}, \ldots, \mathbf{Y}_{i_{p+1}})\right)$, where $(\mathbf{Y}_{i_1}, \mathbf{Y}_{i_2}, \ldots, \mathbf{Y}_{i_{p+1}})$ is a subset of $(\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_M)$. This idea is similar to replacing F_0 by its empirical estimator. By the concept of data depth, some Shewhart, CUSUM, EWMA and other types of charts have been constructed in the literature (cf., Li et al. 2013, Li et al. 2014, Liu 1995). Based on extensive numerical studies, Bush et al. (2010), Stoumbos et al. (2001), Stoumbos and Jones (2000), and some others, have pointed out that control charts based on data depth are generally ineffective, especially in cases when a large reference sample is unavailable. This conclusion is not a surprise because in most charts based on data depth, the nonparametric function F_0 is estimated from a reference sample. While F_0 can be regarded as a parametric function with infinite number of parameters, its empirical estimators would have a large variation, especially in cases when the dimensionality p is relatively large. Consequently, the related charting statistics would have large variation as well, resulting in ineffective detection of process distributional shifts.

Boone and Chakraborti (2012) proposed a MNSPC Shewhart chart based on componentwise signs, described as follows. At the current time point n, assume that we observe a batch of m independent and identically distributed process observations

$$\mathbf{X}_{n1}, \mathbf{X}_{n2}, \ldots, \mathbf{X}_{nm}, \qquad \text{for } n \geq 1,$$

where $\mathbf{X}_{nj} = (X_{nj1}, X_{nj2}, \ldots, X_{njp})'$, for $j = 1, 2, \ldots, m$. Then, for the l-th component, the sign statistic is defined as

$$\xi_{nl} = \sum_{j=1}^{m} \text{sign}(X_{njl} - \widetilde{\mu}_{0l}),$$

where $\text{sign}(u) = -1, 0, 1$, respectively, when $u < 0, = 0, > 0$, and $\widetilde{\boldsymbol{\mu}}_0 = (\widetilde{\mu}_{01}, \widetilde{\mu}_{02}, \ldots, \widetilde{\mu}_{0p})'$ is the IC process median vector. Let $\boldsymbol{\xi}_n = (\xi_{n1}, \xi_{n2}, \ldots, \xi_{np})'$. Then, the Shewhart charting statistic is $\boldsymbol{\xi}_n' \widehat{\Sigma}_{\boldsymbol{\xi}_n} \boldsymbol{\xi}_n$, where $\widehat{\Sigma}_{\boldsymbol{\xi}_n}$ is an estimator of the covariance matrix of $\boldsymbol{\xi}_n$. This chart is difficult to use in cases when $m = 1$ (i.e., a single observation is obtained at each time point), due to the discreteness of its charting statistic. Also, although Boone and Chakraborti (2012) determined the control limits of the Shewhart chart using the χ^2_p distribution, this distribution is appropriate for describing the IC distribution of $\boldsymbol{\xi}_n' \widehat{\Sigma}_{\boldsymbol{\xi}_n} \boldsymbol{\xi}_n$ only when m is large and $\widehat{\Sigma}_{\boldsymbol{\xi}_n}$ is close to $\Sigma_{\boldsymbol{\xi}_n}$. In practice, however, m is usually small (e.g., $m = 5$) and the components of $\boldsymbol{\xi}_n$ are correlated. Thus, the IC distribution of $\boldsymbol{\xi}_n' \widehat{\Sigma}_{\boldsymbol{\xi}_n} \boldsymbol{\xi}_n$ needs to be estimated from an IC dataset using a numerical approach (e.g., a bootstrap algorithm). In that sense, this Shewhart chart is no longer distribution-free. Qiu (2014, section 9.2) generalized it to a multivariate EWMA chart as follows. First, we define

$$\mathbf{E}_n = \lambda \boldsymbol{\xi}_n + (1 - \lambda)\mathbf{E}_{n-1}, \qquad \text{for } n \geq 1, \tag{7}$$

where $\mathbf{E}_0 = \mathbf{0}$ and $\lambda \in (0, 1]$ is a weighting parameter. Then, the chart signals a mean shift when

$$\mathbf{E}_n' \Sigma_{\mathbf{E}_n}^{-1} \mathbf{E}_n > h, \tag{8}$$

where $h > 0$ is a control limit. Boone and Chakraborti (2012) proposed another MNSPC Shewhart chart by generalizing the Wilcoxon signed-rank sum statistic (4) into multivariate cases. In this chart, the signed-rank sum statistic is computed for each component of process observations, and the charting statistic takes a quadratic form of the vector of these componentwise signed-rank sums. A similar MNSPC chart is discussed in Bush et al. (2010). An EWMA chart based on the Wilcoxon signed-rank sum statistic is proposed recently by Chen et al. (2015).

The charts discussed in the previous paragraph use componentwise ordering information that ignores the ordering among different components. To overcome this limitation, Zou and Tsung (2011) and Zou et al. (2012) proposed two EWMA charts using the so-called *spatial sign* and *spatial rank* that were discussed extensively in the nonparametric statistics literature (cf., Oja 2010). Assume that a p-dimensional random vector $\mathbf{X}$ has the mean $\boldsymbol{\mu}_0$. Then, its spatial sign is defined to be $S(\mathbf{X}) = (\mathbf{X} - \boldsymbol{\mu}_0)/\|\mathbf{X} - \boldsymbol{\mu}_0\|$ when $\mathbf{X} \neq \boldsymbol{\mu}_0$, and $\mathbf{0}$ otherwise. For a sample $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ from the distribution of $\mathbf{X}$, the spatial rank of $\mathbf{X}_i$ is defined to be $\mathbf{r}_i = \frac{1}{n}\sum_{j=1}^{n} S(\mathbf{X}_i - \mathbf{X}_j)$, for $i = 1, 2, \ldots, n$. Assume that the IC mean and the IC covariance matrix of $\mathbf{X}$ are $\boldsymbol{\mu}_0$ and Σ_0, respectively, and the "most robust" measure of scatter of the distribution of $\mathbf{X}$ defined by Tyler (1987) is A_0 which is an upper triangular $p \times p$ matrix with positive diagonal elements. Then, the EWMA charting statistic suggested by Zou and Tsung (2011) is defined as $[(2-\lambda)p/\lambda]\mathbf{E}_n'\mathbf{E}_n$, where

$$\mathbf{E}_n = (1-\lambda)\mathbf{E}_{n-1} + \lambda S(A_0\mathbf{X}_n), \qquad \text{for } n \geq 1.$$

Zou and Tsung (2011) pointed out that this chart was distribution-free only in some specially cases and its control limit needed to be estimated from an IC dataset in a general case. A CUSUM chart based on spatial sign was discussed by Li et al. (2013). The EWMA chart suggested by Zou et al. (2012) replaced the spatial sign $S(\mathbf{X}_n)$ in the above expression by a properly defined spatial rank of $\mathbf{X}_n$. Holland and Hawkins (2014) proposed a CPD chart based on spatial rank. That chart can be accomplished using the R-package NPMVCP.

The control charts discussed above are all based on longitudinal ranking of the observed data. Qiu and Hawkins (2001) suggested an alternative strategy for nonparametric SPC based on cross-component ranking of the data. Let $\{\mathbf{X}_n = (X_{n1}, X_{n2}, \ldots, X_{np})', n \geq 1\}$ be the Phase II observations of a p-dimensional process with the IC mean vector $\boldsymbol{\mu}_0$ and the IC covariance matrix Σ_0, and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_p)'$ be the true mean of $\mathbf{X}_n$. Without loss of generality, assume that $\boldsymbol{\mu}_0 = \mathbf{0}$ and $\Sigma_0 = I_{p\times p}$ (otherwise, consider transformed observations $\widetilde{\mathbf{X}}_n = \Sigma_0^{-1/2}(\mathbf{X}_n - \boldsymbol{\mu}_0)$). Qiu and Hawkins noticed that any mean shift violates either $H_0^{(1)}: \mu_1 = \mu_2 = \cdots = \mu_p$ or $H_0^{(2)}: \sum_{j=1}^{p}\mu_j = 0$, where $H_0^{(1)}$ is related to the ranking of the p components of $\mathbf{X}_n$ and $H_0^{(2)}$ is related to their magnitudes. To detect mean shifts violating $H_0^{(1)}$, Qiu and Hawkins suggested using the anti-ranks (or called inverse ranks) of the p components of $\mathbf{X}_n$. The first anti-rank A_{n1} is defined to be the index in $(1, 2, \ldots, p)$ of the smallest component, the last anti-rank A_{np} is the index of the largest component, and so forth. While the p conventional ranks of $(X_{n1}, X_{n2}, \ldots, X_{np})$ are equally important in detecting mean shifts when no prior information is available regarding which component(s) of $\mathbf{X}_n$ would have mean shift, the anti-ranks have the following properties. The first anti-rank is particularly sensitive to downward mean shifts in a small number of components of $\mathbf{X}_n$, and the last anti-ranks is particularly sensitive to upward mean shifts in a small number of components of $\mathbf{X}_n$. If we do not know the direction of a shift, then the combination of the first and last anti-ranks should be sensitive to the shift. In other words, we can reduce the dimension of the SPC problem from p to 2 without losing much efficiency if the anti-ranks are used. Qiu and Hawkins then suggested a nonparametric CUSUM chart based on the anti-ranks for detecting mean shifts violating $H_0^{(1)}$. To detect mean shifts violating $H_0^{(2)}$, a regular CUSUM using $\sum_{j=1}^{p} X_{nj}$ should be effective. However, the above monitoring scheme has an obvious drawback that two separate control charts need to be used for detecting mean shifts violating $H_0^{(1)}$ and $H_0^{(2)}$, respectively, making it inconvenient to use. To overcome this drawback, Qiu and Hawkins (2003) proposed a modification using the anti-ranks of $(X_{n1}, X_{n2}, \ldots, X_{np}, 0)$. This modified CUSUM chart was shown effective in detecting arbitrary mean shifts. From the above description, it can be seen that the IC properties of the nonparametric control charts based on anti-ranks are determined completely by the IC distribution of the anti-ranks. In cases when the IC process distribution is exchangable in the sense that the distribution function is unchanged if the order of measurement components is changed, then the IC distribution of the anti-ranks would be uniform in its domain. Thus, the IC properties of the related control charts would not change in such cases. However, in a general case, parameters in the IC distribution of the anti-ranks should still be estimated from an IC dataset.

In cases when $\mathbf{X}_n$ is multivariate normally distributed, we know that all marginal distributions are normal and the relationship between any two subsets of its p components is linear (i.e., the regression function of one subset on the other is always linear). *In cases when the joint distribution of $\mathbf{X}_n$ is not normal, its marginal distributions and the relationship between a pair of two subsets of the components of $\mathbf{X}_n$ could be complicated, which explains the main reason why multivariate non-Gaussian distributions are difficult to describe. However, if all components of $\mathbf{X}_n$ are categorical, this difficulty disappears because the log-linear modeling approach is effective in describing* *the relationship among categorical variables (e.g., Agresti 2002).* Based on this consideration, Qiu (2008) proposed a general scheme to construct nonparametric multivariate SPC charts, by first categorizing the original components of $\mathbf{X}_n$ (if some components are already categorical, then this step can be skipped for these components), and then describing the joint distribution of the categorized data using a log-linear model. Then, a control chart can be constructed accordingly by comparing the empirical and IC distributions of the categorical data, as in univariate cases discussed in Section 3. In the chart by Qiu (2008), only two categories (i.e., $\geq$ or $<$ the IC median) are used for each continous component of $\mathbf{X}_n$, for simplicity. Its performance might be improved if more categories are used. Like other multivariate nonparametric control charts discussed above, the control charts based on data categorization are distribution-free only in some special cases (e.g., the process distribution is exchangable and symmetric). In a general case, some parameters for describing the IC distribution of the categorized data should be estimated from an IC dataset by the log-linear modeling.

Next, we present a numerical study to compare several representative MNSPC charts described above. The MNSPC charts considered here include (i) the EWMA chart (7)-(8) based on the componentwise sign statistic $\boldsymbol{\xi}_n$, denoted as SIGN-EWMA, (ii) the CPD chart by Holland and Hawkins (2014) that is based on the spatial rank, denoted as SR-CPD, (iii) the modified anti-rank-based chart by Qiu and Hawkins (2003), denoted as ANTIRANK, and (iv) the control chart based on data categorization that was proposed by Qiu (2008), denoted as CATEGORIZE. In SR-CPD, the control limits provided in Holland and Hawkins (2014) are used, where a "burn-in" sample of 32 IC observations is used. In ANTIRANK, only the first and last anti-ranks are considered. These four charts are chosen mainly because of the following three considerations: 1) they represent four types of charts that use the sign statistic, spatial rank, cross-component anti-rank, and data categorization, respectively, 2) all of them are either EWMA, CPD, or CUSUM charts that are good at detecting persistent mean shifts, and 3) we have their software.

In the numerical study, we assume that there is a single observation at each time point, observations at different time points are independent, the IC distribution is known, $p = 5$, and $ARL_0 = 200$. So, the impact of the estimation error of the IC distribution from an IC data and the data autocorrelation are not considered here. However, it should be pointed out that one important property of the CPD chart SR-CPD is that it does not need any prior information about the IC process distribution, besides a "burn-in" sample, in order to run that chart. Thus, by assuming the IC distribution to be known, we may have put it in a disadvantageous status in the comparison. To account for this, besides the original CPD chart SR-CPD, we also consider the SR-CPD chart that uses a large "burn-in" sample of 532 observations, denoted as SR-CPD1. To use a large "burn-in" sample, the SR-CPD1 chart could learn enough information about the IC distribution from the "burn-in" sample. The following four IC distributions are considered:

$IndT_5(3)$: The 5 components of $\mathbf{X}_n$ are independent and each of them has the standardized version with mean 0 and variance 1 of a t_3 distribution;

$IndChisq_5(3)$: The 5 components of $\mathbf{X}_n$ are independent and each of them has the standardized version with mean 0 and variance 1 of a χ^2_3 distribution;

$Normal_5(\mathbf{0}, \Sigma_{CS05})$: $\mathbf{X}_n \sim N_5(\mathbf{0}, \Sigma_{CS05})$, where Σ_{CS05} is a covariance matrix with all diagonal elements being 1 and all off-diagonal elements being 0.5 (i.e., a compound symmetry case);

$Mixture$: $X_{n1} \sim N(0,1)$, X_{n2} has the standardized version with mean 0 and variance 1 of a t_3 distribution, X_{n3} has the standardized version with mean 0 and variance 1 of a χ^2_3 distribution, $X_{n4} \sim (X_{n2}+\eta_1)/\sqrt{2}$, and $X_{n5} \sim (X_{n3}+\eta_2)/\sqrt{2}$, where η_1 and η_2 are two independent random numbers generated from $N(0,1)$.

The first two cases $IndT_5(3)$ and $IndChisq_5(3)$ represent different scenarios when the components of $\mathbf{X}_n$ are indendent, while the last two cases represent scenarios when they are correlated. At time point τ, assume that the process mean has one of the following five shifts:

$$M1 : (-0.5, 0, 0, 0, 0), \quad M2 : (-1, 0, 0, 0, 0), \quad M3 : (-1, 1, 0, 0, 0),$$
$$M4 : (-0.5, -0.5, -0.5, 0.5, 0.5), \quad M5 : (0.5, 0.5, 0.5, 0.5, 0.5).$$

Because the SR-CPD chart requires a "burn-in" sample and Holland and Hawkins (2014) recommended the size of that sample to be 33 or more, we first consider the case when $\tau = 50$. To use the four control charts for process monitoring, we need to choose their procedure parameters (e.g., λ and h in SIGN-EWMA) properly. To make the comparison among the four charts as fair as possible, in this paper we choose the parameters of each chart so that (i) the actual $ARL_0 = 200$ and (ii) the ARL_1 value reaches the minimum when detecting a given shift. Namely, we compare the optimal performace of the four charts in this study, as we did in univariate cases (cf., Figure 4).

Based on 10,000 replicated simulations, the calculated ARL_1 values of the five charts are shown in Figure 5 in the log scale, along with their actual ARL_0 values. From the plot, we can see that (i) all five methods are reliable to use in the four cases considered in the sense that their actual ARL_0 values are close to the specified value 200, (ii) no single method is unanimously better than the remaining methods, (iii) the chart SR-CPD1 is unanimously better than SR-CPD, as expected, and it is also better than the other charts in most cases, and (iv) it seems that the charts SR-CPD1, ANTIRANK and CATEGORIZE are better in detecting small shifts (e.g. M1) when the components of $\mathbf{X}_n$ are correlated (cf., plots (c) and (d)). The corresponding results when $\tau = 0$ (i.e., zero-state run length performance) are shown in Figure 6. The charts SR-CPD and SR-CPD1 are not included in such cases because their requirement of a "burn-in" sample cannot be satisfied. From the plots, it can be seen that (i) the chart SIGN-EWMA is unanimously worse than the other two charts, (ii) the two charts ANTIRANK and CATEGORIZE perform similarly well, and (iii) it seems that CATEGORIZE performs slightly better than ANTIRANK when detecting small shifts (i.e., $M1$). We also tried cases when τ is between 0 and 50, and the results are between those shown in Figure 5 and Figure 6.

Although there have been some MNSPC charts proposed in the literature, as discussed above, there are still some fundamental issues related to the MNSPC problem that need to be addressed. For instance, most existing MNSPC methods are based on either the longitudinal ranking or the cross-component ranking information in the observed data for process monitoring. Intuitively, a more efficient solution to the MNSPC problem is to combine the longitudinal ranking and cross-component ranking information in the observed data. But, it is still unknown to us how to combine the two types of ranking information effectively. In the method based on data categorization, it has been shown in Qiu and Li (2011b) that its performance can be improved if more than 2 categories are used in univariate cases, and some practical guidelines about the selection of the number of categories are given in the second last paragraph in Section 3. In multivariate cases, it is unknown whether similar guidelines are still valid. It has been pointed out several times in the paper that some information in the observed data will be lost by using either the ranking information or the data categorization. This is especially true in multivariate cases. So, a natural research question is how to minimize the lost information in MNSPC. All these research questions require much future research effort.

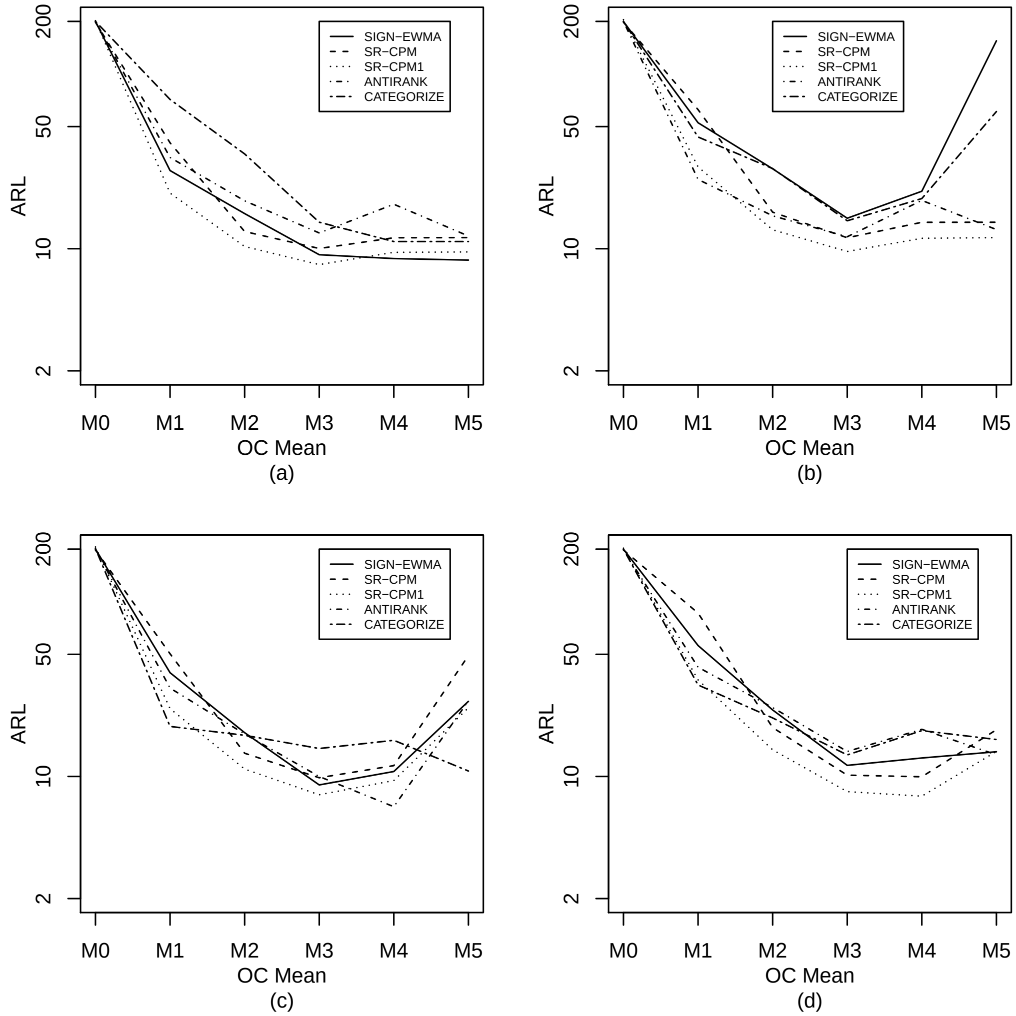

Figure 5: Calculated ARL_0 and ARL_1 values of the four MNSPC charts for detecting shifts M0-M5 when $\tau = 50$, where M0=(0,0,0,0,0). (a) $IndT_5(3)$; (b) $IndChisq_5(3)$; (c) $Normal_5(\mathbf{0}, \Sigma_{CS05})$; and (d) $Mixture$. The $y-$axes are in the log scale.

5 Some Concluding Remarks

In this big data era, SPC has found more and more applications in engineering, environmental research, medicine, public health, and other disciplines and areas (Qiu 2017, 2018), because data streams are common in these applications and it is often a fundamental task to online monitor the properties of the related data streams. When the data structure becomes more complicated, conventional SPC charts need to be modified or generalized properly in order to be still useful. In this paper, we only focus on cases when the conventional normality assumption is invalid. In practice, other assumptions, such as the independent observations and the identical IC distribution, could all be violated. For instance, when monitoring the spatial-temporal pattern of the incidence

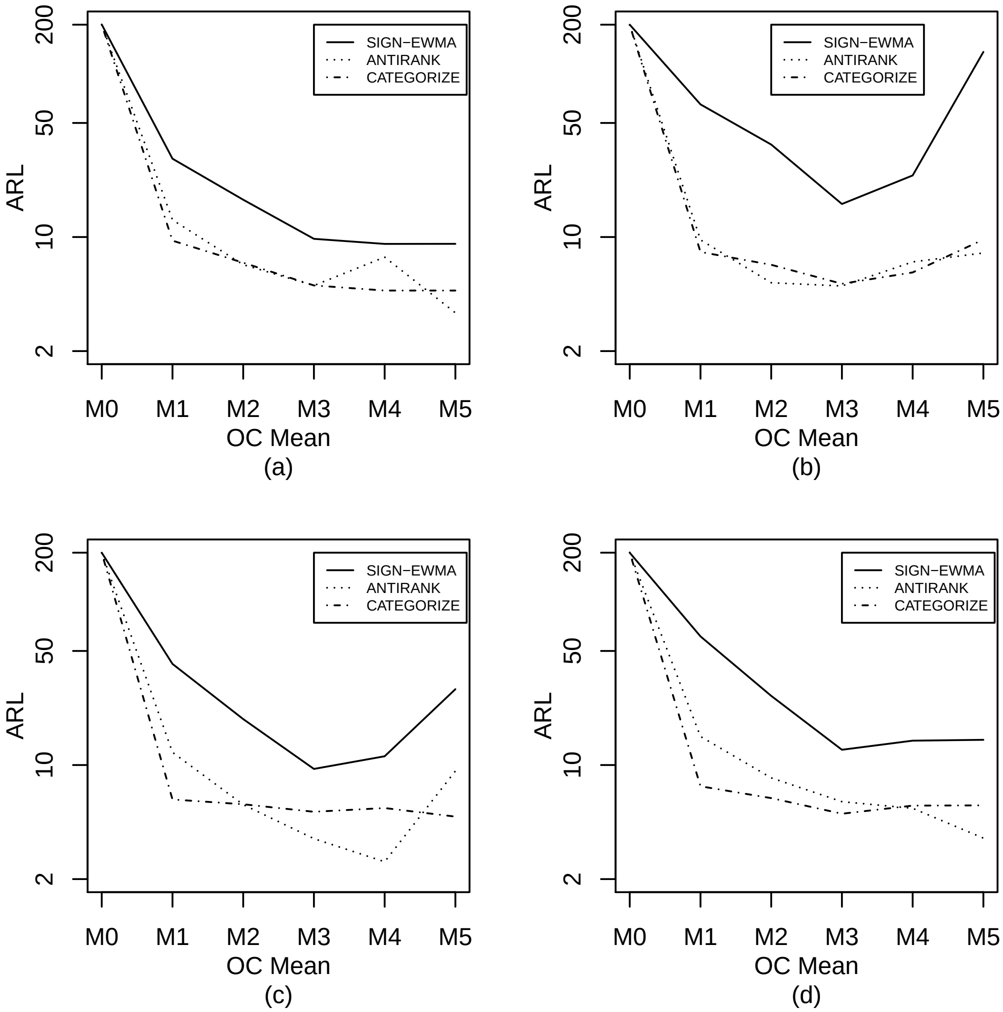

Figure 6: Calculated ARL_0 and ARL_1 values of the three MNSPC charts for detecting shifts M0-M5 when $\tau = 0$, where M0=(0,0,0,0,0). (a) $IndT_5(3)$; (b) $IndChisq_5(3)$; (c) $Normal_5(\mathbf{0}, \Sigma_{CS05})$; and (d) $Mixture$. The $y-$axes are in the log scale.

rate of an infectious disease across US and over time, observed incidence rates could be spatially and temporally correlated. Furthermore, even when there is no disease outbreak in a given region and within a given time period, the distribution of disease incidence rate could still change in the given region and the given time period, due to seasonality and other environmental factors (Zhang et al. 2015). Therefore, new SPC methods are needed for such applications, and the new methods should be able to accommodate the complicated data structure properly.

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