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SUPPLEMENTARY MATERIALS

Supplementary Materials for Nonparametric Estimation of the Spatio-Temporal Covariance Structure

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Abstract

To save some space in the paper with the above title, the proofs of the theorems are presented in this supplementary file.

PROOF OF THEOREM 1

For simplicity of expression, we use y_{ij} and ε_{ij} to denote $y(t_i, \mathbf{s}_{ij})$ and $\varepsilon(t_i, \mathbf{s}_{ij})$, respectively. At a given point $(t, \mathbf{s}) \in [0,1] \times \Omega$, from (4), we have

$$\begin{aligned}\widehat{\lambda}(t,\mathbf{s}) &= \boldsymbol{e}_1^T(\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}_0\mathbf{Y} \\ &= \boldsymbol{e}_1^T(\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}_0\lambda + \boldsymbol{e}_1^T(\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}_0\boldsymbol{\varepsilon} \\ &= \Pi_1 + \Pi_2,\end{aligned} \tag{A.1}$$

where $\boldsymbol{\lambda} = E(\mathbf{Y})$ and $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \ldots, \varepsilon_{1m_1}, \ldots, \varepsilon_{nm_n})^T$. For Π_1, by the Taylor's expansion, it can be shown that

$$\Pi_1 = \boldsymbol{e}_1^T(\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}_0(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\mathcal{R}}) = \lambda(t,\mathbf{s}) + \Pi_3, \tag{A.2}$$

where $\boldsymbol{\beta} = (\lambda(t,\mathbf{s}), \partial\lambda(t,\mathbf{s})/\partial t, \partial\lambda(t,\mathbf{s})/\partial\mathbf{s})^T$, $\boldsymbol{\mathcal{R}} = (r_{11}, \ldots, r_{1m_1}, \ldots, r_{nm_n})^T$, $r_{ij} = ((t_i - t), (\mathbf{s}_{ij} - \mathbf{s})^T)\,\boldsymbol{\mathcal{H}}(t'_{ij}, \mathbf{s}'_{ij})((t_i - t), (\mathbf{s}_{ij} - \mathbf{s})^T)^T$, $\boldsymbol{\mathcal{H}}$ is the Hessian matrix of $\lambda(t,\mathbf{s})$, and $t'_{ij} \in [0,1]$, $\mathbf{s}'_{ij} \in \Omega$, for $j = 1, \ldots, m_i, i = 1, \ldots, n$. From (A.1) and (A.2), we have $\widehat{\lambda}(t,\mathbf{s}) = \lambda(t,\mathbf{s}) + \Pi_2 + \Pi_3$. For Π_2, it can be checked that

$$\begin{aligned}\Pi_2 &= \boldsymbol{e}_1^T(\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathbf{X}^T\mathbf{W}_0\boldsymbol{\varepsilon} \\ &= \boldsymbol{e}_1^T(\mathcal{D}(m,n)\mathbf{X}^T\mathbf{W}_0\mathbf{X})^{-1}\mathcal{D}(m,n)\mathbf{X}^T\mathbf{W}_0\boldsymbol{\varepsilon} \\ &= \boldsymbol{e}_1^T\mathbf{A}^{-1}(t,\mathbf{s})\mathbf{B}(t,\mathbf{s}),\end{aligned} \tag{A.3}$$

where $\mathcal{D}(m,n) = (nh_1mh_2^2f(\mathbf{s}))^{-1}$, $\mathbf{A}(t,\mathbf{s}) = \mathcal{D}(n,m)\mathbf{X}^T\mathbf{W}_0\mathbf{X}$ is a 4×4 matrix and $\mathbf{B}(t,\mathbf{s}) = \mathcal{D}(n,m)\mathbf{X}^T\mathbf{W}_0\boldsymbol{\varepsilon}$ is a vector of length 4. Next, we consider the first element of the vector $\mathbf{B}(t,\mathbf{s})$, i.e.,

$$\mathbf{B}_1(t,\mathbf{s}) = (nh_1)^{-1}\sum_{i=1}^n K_1\left((t_i - t)/h_1\right)\varepsilon_i(\mathbf{s}), \tag{A.4}$$

where $\varepsilon_i(s) = \{mh_2^2f(\mathbf{s})\}^{-1}\sum_{j=1}^{m_i} K_2\left(d_E(\mathbf{s}_{ij},\mathbf{s})/h_2\right)\varepsilon_{ij}$. We will show below that $\mathbf{B}_1(t,\mathbf{s}) = O_p(h_1^2 + h_2^2 + \{1/(nh_1)\}^{1/2})$.

For the variance of $\mathbf{B}_1(t,\mathbf{s})$ defined in (A.4), it is clear that

$$\mathrm{Var}\left(\mathbf{B}_1(t,\mathbf{s})\right) = \{nh_1\}^{-2}\sum_{i=1}^n\sum_{k=1}^n K_1\left((t_i - t)/h_1\right)K_1\left((t_k - t)/h_1\right)\mathrm{Cov}\left(\varepsilon_i(\mathbf{s}), \varepsilon_k(\mathbf{s})\right). \tag{A.5}$$

To calculate the covariance Cov $\left(\varepsilon_i(\boldsymbol{s}), \varepsilon_k(\boldsymbol{s})\right)$, let $\mathcal{S}_\sigma$ be the σ-algebra generated by $\{\boldsymbol{s}_{ij}, j = 1, \ldots, m_i, i = 1, \ldots, n\}$. Given any possible values of $\mathcal{S} = \{\boldsymbol{s}_{ij}, j = 1, \ldots, m_i, i = 1, \ldots, n\}$, by the Davydov's inequality, it can be checked that

$$\begin{aligned} E\left(\varepsilon_i(\boldsymbol{s})\varepsilon_k(\boldsymbol{s})|\mathcal{S}\right) =& \{mh_2^2 f(\boldsymbol{s})\}^{-2} \sum_{j=1}^{m_i}\sum_{l=1}^{m_k} K_2\left(\frac{d_E(\boldsymbol{s}_{ij}, \boldsymbol{s})}{h_2}\right) K_2\left(\frac{d_E(\boldsymbol{s}_{kl}, \boldsymbol{s})}{h_2}\right) \mathrm{Cov}(\varepsilon_{ij}, \varepsilon_{ij}) \\ \leq& 12\{mh_2^2 f(\boldsymbol{s})\}^{-2} C_\varepsilon^{2/\delta} C_0^{(\delta-2)/\delta} \sum_{j=1}^{m_i}\sum_{l=1}^{m_k} K_2\left(\frac{d_E(\boldsymbol{s}_{ij}, \boldsymbol{s})}{h_2}\right) K_2\left(\frac{d_E(\boldsymbol{s}_{kl}, \boldsymbol{s})}{h_2}\right) \\ &\times \exp\left\{-\frac{C_1(\delta-2)}{\delta}|k-i|\right\} \\ =& D^2(\boldsymbol{s}) \exp\left\{-\frac{C_1(\delta-2)}{\delta}|k-i|\right\} \times O(1), \end{aligned} \tag{A.6}$$

where $D(\boldsymbol{s}) = \max_{1\leq i\leq n} D_i(\boldsymbol{s})$ and $D_i(\boldsymbol{s}) = \{mh_2^2 f(\boldsymbol{s})\}^{-1} \sum_{j=1}^{m_i} K_2\left(d_E(\boldsymbol{s}_{ij}, \boldsymbol{s})/h_2\right)$. Denote $C_{\min} = \min_{1\leq i\leq n} m_i/m$ and $C_{\max} = \max_{1\leq i\leq n} m_i/m$. Note that the random variable $D_i(\boldsymbol{s})$ is nonnegative, by the Bernstein's inequality, it can be shown that

$$\begin{aligned} & E\left(D(\boldsymbol{s})^2\right) = E\Big[\big(\max_{1\leq i\leq n} D_i(\boldsymbol{s})\big)^2\Big] \\ =& E\Big[\max_{1\leq i\leq n} D_i^2(\boldsymbol{s})\Big] \leq \sum_{k=0}^{\infty}(k+1)\mathrm{Pr}\Big(\max_{1\leq i\leq n} D_i^2(\boldsymbol{s}) \geq k\Big) \\ \leq& 1 + n\sum_{k=1}^{\infty}(k+1)\max_{1\leq i\leq n}\mathrm{Pr}\Big(D_i(\boldsymbol{s}) \geq \sqrt{k}\Big) \\ \leq& 1 + n\sum_{k=1}^{\infty}(k+1)\exp\left\{-mh_2^2\left(C_{\min}\left(\sqrt{k} - C_{\max}\right)\right)\right\} \\ =& O(1) + O(1) \times \sum_{k=1}^{\infty}(k+1)\exp\left\{-mh_2^2\left(C_{\min}\left(\sqrt{k} - C_{\max}\right) - 1\right)\right\} = O(1). \end{aligned} \tag{A.7}$$

It follows from (A.6) and (A.7) that

$$\begin{aligned} & \mathrm{Cov}\left(\varepsilon_i(\boldsymbol{s}), \varepsilon_k(\boldsymbol{s})\right) = E\left[E\left(\varepsilon_i(\boldsymbol{s})\varepsilon_k(\boldsymbol{s})|\mathcal{S}\right)\right] \\ =& E\left(D^2(\boldsymbol{s})\right)\exp\left\{-\frac{C_1(\delta-2)}{\delta}|k-i|\right\} \times O(1) \\ =& \exp\left\{-\frac{C_1(\delta-2)}{\delta}|k-i|\right\} \times O(1). \end{aligned} \tag{A.8}$$

Then by (A.5), we have

$$\begin{aligned} \mathrm{Var}(\mathbf{B}_1(t, \boldsymbol{s})) =& O(1) \times \{nh_1\}^{-2} \sum_{i=1}^{n}\sum_{k=1}^{n} K_1\left((t_i - t)/h_1\right) K_1\left((t_k - t)/h_1\right) \exp\left\{-\frac{C_1(\delta-2)}{\delta}|k-i|\right\} \\ =& O(1) \times \{nh_1\}^{-2} \sum_{i=1}^{n}\sum_{\tau=0}^{\infty} K_1\left((t_i - t)/h_1\right) \exp\left\{-\frac{C_1(\delta-2)}{\delta}\tau\right\} \\ =& O(1) \times \{nh_1\}^{-1} \sum_{\tau=0}^{\infty} \exp\left\{-\frac{C_1(\delta-2)}{\delta}\tau\right\} = O(1/(nh_1)). \end{aligned} \tag{A.9}$$

Note that $E(\mathbf{B}_1(t, \boldsymbol{s})) = 0$, we have

$$\mathbf{B}_1(t, \boldsymbol{s}) = O_p(\{1/(nh_1)\}^{1/2}) = O_p(\nu(n, m)), \tag{A.10}$$

where $\nu(n, m) = h_1^2 + h_2^2 + \{1/(nh_1)\}^{1/2}$. After defining $\nu^*(n, m) = \nu(n, m) + \{1/(mh_2^2)\}^{1/2}$ and $\boldsymbol{\mu}_2(K) = \mathrm{diag}\{\mu_{21}(K), \mu_{22}(K), \mu_{22}(K)\}$, where $\mu_{21}(K) = \int x^2 K_1(x)dx$, $\mu_{22}(K) = \int u_1^2 K_2(d_E(\boldsymbol{u}, \boldsymbol{0}))d\boldsymbol{u}$, and $\boldsymbol{u} = (u_1, u_2)^T$. It can be shown similarly that

$$\mathbf{A}(t, \boldsymbol{s}) = \begin{pmatrix} a + O_p(\nu^*(n, m)) & \mathbf{1}^T \boldsymbol{H} O_p(\nu^*(n, m)) \\ \boldsymbol{H}\mathbf{1} O_p(\nu^*(n, m)) & \boldsymbol{C}(1 + O_p(\nu^*(n, m))) \end{pmatrix}, \quad \text{and}$$

$$\mathbf{B}(t, \boldsymbol{s}) = \begin{pmatrix} O_p(\nu(n, m)) \\ \boldsymbol{H}\mathbf{1} O_p(\nu(n, m)) \end{pmatrix},$$

where $a \in [C_{\min}, C_{\max}]$, and all elements of the 3×3 matrix $\boldsymbol{C}$ are in the same order of the corresponding elements of $\boldsymbol{H}^2 \boldsymbol{\mu}_2(K)$ $\mathbf{1} = (1,1,1)^T$ and $\boldsymbol{H} = \mathrm{diag}\{h_1, h_2, h_2\}$. It follows that

$$\Pi_2 = e_1^T \mathbf{A}^{-1}(t,\boldsymbol{s})\mathbf{B}(t,\boldsymbol{s}) = O_p(\nu(n,m)). \tag{A.11}$$

By using similar arguments, we have $\Pi_3 = O_p(\nu(n,m))$. By combining this result with (A.1), (A.2) and (A.11), the result in (11) of the paper is true.

PROOF OF THEOREM 2

From (A.1)-(A.3), we have $\widehat{\lambda}(t,\boldsymbol{s}) = \lambda(t,\boldsymbol{s}) + \Pi_2 + \Pi_3$, where $\Pi_2 = e_1^T \mathbf{A}^{-1}(t,\boldsymbol{s})\mathbf{B}(t,\boldsymbol{s})$, and $\mathbf{A}^{-1}(t,\boldsymbol{s})$ and $\mathbf{B}(t,\boldsymbol{s})$ are defined in the proof of Theorem 1. Next we will show that $\mathbf{B}_1(t,\boldsymbol{s}) = O_p(a(n,m))$ uniformly for $(t,\boldsymbol{s}) \in [0,1]\times\Omega$, where $a(n,m) = \{\log^2(n)/(nh_1^2)\}^{1/2}$.

First, note that the spatial location of interest Ω is bounded, then it is clear that $[0,1]\times\Omega$ can be covered by $N^* = O(\{a(n,m)h_1\}^{-3})$ regions $\{R_l, l = 1,...,\widetilde{N}\}$, where $R_l = \{(t,\boldsymbol{s}) : |t - t_l^*| \le a(n,m)h_1, \ d_E(\boldsymbol{s},\boldsymbol{s}_l^*) \le a(n,m)h_1\}$ and $\{(t_l^*, \boldsymbol{s}_l^*), l = 1,\dots,N^*\}$ are the centroids of the N^* regions. Since both kernel functions $K_1(x)$ and $K_2(x)$ are Lipschitz-1 continuous, let $0 < L_K < \infty$ be their Lipschitz constant. Because it is assumed that $h_1/h_2 = O(1)$, we can find some constant $C_2 > 0$ such that $h_1 \le C_2 h_2$. Define $C_K = \sup_{x\in\mathbb{R}}\{K_1(x), K_2(x)\}$. Then, for any $(t,\boldsymbol{s}) \in R_l$ and a sufficiently large n, we have

$$\begin{aligned}&\left|K_1\left(\frac{t_i - t}{h_1}\right)K_2\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s})}{h_2}\right) - K_1\left(\frac{t_i - t_l^*}{h_1}\right)K_2\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s}_l^*)}{h_2}\right)\right| \\ \le& C_K L_K h_1^{-1}\{|t - t_l^*| + C_2 d_E(\boldsymbol{s},\boldsymbol{s}_l^*)\}\mathrm{I}\left(\frac{|t_i - t_l^*|}{h_1} \le 2L_1\right)\mathrm{I}\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s}_l^*)}{h_2} \le 2L_2\right),\end{aligned} \tag{A.12}$$

where $[-L_1, L_1]$ and $[-L_2, L_2]$ are the finite supports for $K_1(x)$ and $K_2(x)$, respectively. Define $\widetilde{K}_1(x) = 1/(2L_1)\mathrm{I}(|x| \le 2L_1)$ and $\widetilde{K}_2(x) = 1/(4\pi L_2^2)\mathrm{I}(|x| \le 2L_2)$. Then, by (A.12), there exists a constant $C_3 > 0$ such that

$$\begin{aligned}&\left|K_1\left(\frac{t_i - t}{h_1}\right)K_2\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s})}{h_2}\right) - K_1\left(\frac{t_i - t_l^*}{h_1}\right)K_2\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s}_l^*)}{h_2}\right)\right| \\ &\qquad \le C_3 a(m,n)\widetilde{K}_1\left(\frac{t_i - t_l^*}{h_1}\right)\widetilde{K}_2\left(\frac{d_E(\boldsymbol{s}_{ij},\boldsymbol{s}_l^*)}{h_2}\right).\end{aligned} \tag{A.13}$$

Define

$$\widetilde{\mathbf{B}}_1(t,\boldsymbol{s}) = \{nmh_1h_2^2 f(\boldsymbol{s})\}^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\widetilde{K}_1\left((t_i - t)/h_1\right)\widetilde{K}_2\left(d_E(\boldsymbol{s}_{ij},\boldsymbol{s})/h_2\right)|\varepsilon_{ij}|.$$

Since $\widetilde{K}_1(\cdot)$ and $\widetilde{K}_2(\cdot)$ satisfy the assumptions about the kernel function in Theorem 1, it can be checked that

$$E\left(\widetilde{\mathbf{B}}_1(t,\boldsymbol{s})\right) \le C_\varepsilon^{1/\delta}\left(1 + O(h_2^2 + 1/(nh_1))\right) < \infty,$$

where δ and C_ε are defined in Theorem 1. Based on the result in (A.13), it can be checked that

$$\begin{aligned}\sup_{(t,\boldsymbol{s})\in R_l}\left|\mathbf{B}_1(t,\boldsymbol{s}) - E\{\mathbf{B}_1(t,\boldsymbol{s})\}\right| &\le \left|\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*)\}\right| + C_3 a(n,m)\left[\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*) + E\{\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*)\}\right] \\ \le \left|\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*)\}\right| &+ C_3 a(n,m)\left[|\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*)\}|\right] + 2C_3 a(n,m)E\{\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*)\} \\ \le \left|\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*)\}\right| &+ C_3\left[|\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*)\}|\right] + 2C_3 a(n,m)T,\end{aligned} \tag{A.14}$$

where the final inequality is obtained because $a(n,m) < 1$ and $T > E(\widetilde{\mathbf{B}}_1(t,\boldsymbol{s}))$ when n, m and T are large enough. By (A.14), it can be checked that

$$\begin{aligned}&\Pr\left(\sup_{(t,\boldsymbol{s})\in[0,1]\times\Omega}|\mathbf{B}_1(t,\boldsymbol{s}) - E\{\mathbf{B}_1(t,\boldsymbol{s})\}| > (2 + 4C_3)Ta(n,m)\right) \\ \le& N^*\max_{1\le l\le N^*}\Pr\left(|\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\mathbf{B}_1(t_l^*,\boldsymbol{s}_l^*)\}| > 2Ta(n,m)\right) \\ &+ N^*\max_{1\le l\le N^*}\Pr\left(|\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*) - E\{\widetilde{\mathbf{B}}_1(t_l^*,\boldsymbol{s}_l^*)\}| > 2Ta(n,m)\right).\end{aligned} \tag{A.15}$$

For the two parts on the right-hand side of (A.15), we can use similar arguments to find their upper bounds, because both $(K_1(x), K_2(x))$ and $(\widetilde{K}_1(x), \widetilde{K}_2(x))$ satisfy the assumptions on the kernel functions given in Theorem 1.

Second, for any $(t, \boldsymbol{s}) \in [0,1] \times \Omega$, by the fact that $E\{\mathbf{B}_1(t, \boldsymbol{s})\} = 0$, we have

$$\Pr\left(|\mathbf{B}_1(t, \boldsymbol{s}) - E\{\mathbf{B}_1(t, \boldsymbol{s})\}| > 2Ta(n, m)\right) = \Pr\left(|\mathbf{B}_1(t, \boldsymbol{s})| > 2Ta(n, m)\right). \tag{A.16}$$

Next, we replace $\varepsilon_i(\boldsymbol{s})$ in $\mathbf{B}_1(t, \boldsymbol{s})$ by its truncated version $\varepsilon_i(\boldsymbol{s})\mathrm{I}(|\varepsilon_i(\boldsymbol{s})| \le \varphi_n)$, and evaluate the error caused by this truncation, where $\varphi_n = \{n/\log(n)^2\}^{1/2}$ and $\varepsilon_i(\boldsymbol{s})$ is defined in (A.4). Based on the assumption that $E|\varepsilon(t_i, \boldsymbol{s}_{ij})|^\delta \le C_\varepsilon$ for some $\delta > 5$ and all (i, j), it can be checked that $E\left(|\varepsilon_i(\boldsymbol{s})|^5\right) \le \Theta_0$, for some constant $\Theta_0 > 0$. Define

$$\mathrm{TR}(t, \boldsymbol{s}) = \frac{1}{nh_1} \sum_{i=1}^{n} K_1\left((t_i - t)/h_1\right) \varepsilon_i(\boldsymbol{s}) \mathrm{I}\left(|\varepsilon_i(\boldsymbol{s})| > \varphi_n\right),$$

then we have

$$\begin{aligned} E\left(|\mathrm{TR}(t, \boldsymbol{s})|\right) &\le \frac{1}{nh_1} \sum_{i=1}^{n} K_1\left((t_i - t)/h_1\right) E\left\{|\varepsilon_i(\boldsymbol{s})| \mathrm{I}\left(|\varepsilon_i(\boldsymbol{s})| > \varphi_n\right)\right\} \\ &\le \frac{\Theta_0}{nh_1} \sum_{i=1}^{n} K_1\left((t_i - t)/h_1\right) \varphi_n^{-4} = O(\varphi_n^{-4}). \end{aligned} \tag{A.17}$$

By the Markov's inequality, it is clear that $|\mathrm{TR}(t, \boldsymbol{s}) - E\left(\mathrm{TR}(t, \boldsymbol{s})\right)| = O_p(\varphi_n^{-4})$, for any $(t, \boldsymbol{s}) \in [0,1] \times \Omega$.

Let $\widetilde{\varepsilon}_i(t, \boldsymbol{s}) = \varepsilon_i(\boldsymbol{s})\mathrm{I}(|\varepsilon_i(\boldsymbol{s})| \le \varphi_n) K_1((t_i - t)/h_1)$, $Z_i(t, \boldsymbol{s}) = \widetilde{\varepsilon}_i(t, \boldsymbol{s}) - E\{\widetilde{\varepsilon}_i(t, \boldsymbol{s})\}$ and $\mathbf{B}_1^*(t, \boldsymbol{s}) = \mathbf{B}_1(t, \boldsymbol{s}) - \mathrm{TR}(t, \boldsymbol{s})$. It can be checked that $\mathbf{B}_1^*(t, \boldsymbol{s}) - E\{\mathbf{B}_1^*(t, \boldsymbol{s})\} = (nh_1)^{-1} \sum_{i=1}^n Z_i(t, \boldsymbol{s})$, for $(t, \boldsymbol{s}) \in [0,1] \times \Omega$. Given any possible values of $\mathcal{S} = \{\boldsymbol{s}_{ij}, j = 1, \dots, m_i, i = 1, \dots, n\}$, we have $|Z_i(t, \boldsymbol{s})| \le 2C_K \varphi_n$, and for any positive integer $L \le n$, it can be checked that

$$E\Big[\sum_{i=1}^{L} Z_i(t, \boldsymbol{s})^2 \mid \mathcal{S}_\sigma\Big] \le \Theta_1 L D(\boldsymbol{s})^2,$$

for some constant $\Theta_1 > 0$, where $\mathcal{S}_\sigma$ is the σ-algebra generated by the spatial locations $\mathcal{S} = \{\boldsymbol{s}_{11}, \dots, \boldsymbol{s}_{nm_n}\}$, $D(\boldsymbol{s}) = \max_{1 \le i \le n} D_i(\boldsymbol{s})$, and $D_i(\boldsymbol{s}) = (mh_2^2 f(\boldsymbol{s}))^{-1} \sum_{j=1}^{m_i} K_2(d_E(\boldsymbol{s}_{ij}, \boldsymbol{s})/h_2)$. Note that $\{\boldsymbol{s}_{ij}, j = 1, \dots, m_i, i = 1, \dots, n\}$ are independent of the random errors $\{\varepsilon_{11}, \dots, \varepsilon_{nm_n}\}$. So, for given $\{\boldsymbol{s}_{ij}, j = 1, \dots, m_i, i = 1, ..., n\}$, $\{Z_i(t, \boldsymbol{s}), i = 1, \dots, n\}$ is a strong mixing sequence with the strong mixing coefficient $\{\widetilde{\alpha}(k), k = 0, 1, \dots\}$, and $\widetilde{\alpha}(k) \le \alpha(k)$, for $k \ge 1$. Let L_n be an integer closest to $\{T^{1/2}\log(n)\}/(10C_K)$. Then, we have $nh_1 a(n, m)T > 8C_K L_n \varphi_n$ when n is large. By Theorem 2.1 in Liebscher (1996), for $1 \le l \le N^*$, it can be shown that

$$\begin{aligned} &\Pr\left(|\mathbf{B}_1^*(t_l^*, \boldsymbol{s}_l^*) - E(\mathbf{B}_1^*(t_l^*, \boldsymbol{s}_l^*))| > a(n, m)T \mid \mathcal{S}_\sigma\right) \\ \le& 4\exp\left(-\frac{T^2 \log(n)}{64\Theta_1 D(\boldsymbol{s})^2 + T^{3/2}}\right) + \frac{40nC_0 C_K}{T^{1/2}\log(n)} \exp\left(-\frac{C_1 T^{1/2}\log(n)}{10C_K}\right), \end{aligned} \tag{A.18}$$

when $\log(n) > 1$. Note that the second term on the right-hand side of (A.18) is independent of the choice of $\{\boldsymbol{s}_{ij}, j = 1, \dots, m_i, i = 1, ..., n\}$. Then, by the Bernstein's inequality, we have

$$\begin{aligned} &\Pr\left(|\mathbf{B}_1^*(t_l^*, \boldsymbol{s}_l^*) - E(\mathbf{B}_1^*(t_l^*, \boldsymbol{s}_l^*))| > a(n, m)T\right) \\ \le& E\left\{4\exp\left(-\frac{T^2\log(n)}{64\Theta_1 D(\boldsymbol{s})^2 + T^{3/2}}\right) \mathrm{I}(D(\boldsymbol{s}) \le C_{\max} T^{1/2})\right\} \\ &+ 4\Pr\left(D(\boldsymbol{s}) > C_{\max} T^{1/2}\right) + \frac{40nC_K}{T^{1/2}\log(n)} \exp\left(-\frac{C_1 T^{1/2}\log(n)}{10C_K}\right) \\ =& 4\exp\left(-\frac{T^2\log(n)}{64\Theta_1 C_{\max}^2 T + T^{3/2}}\right) + O\left(n\exp\left(-mh_2^2\left(C_{\min} T^{1/2} - 1\right)\right)\right) \\ &+ \frac{40nC_K}{T^{1/2}\log(n)} \exp\left(-\frac{C_1 T^{1/2}\log(n)}{10C_K}\right), \end{aligned} \tag{A.19}$$

where $C_{\max}$ and $C_{\min}$ are defined in the proof of Theorem 1. In addition, from (A.17), by the Markov's inequality, we have

$$\Pr\left(|\mathrm{TR}(t_l^*, \boldsymbol{s}_l^*) - E\{\mathrm{TR}(t_l^*, \boldsymbol{s}_l^*)\}| > a(n, m)T\right) = O\left(\{a(n, m)T\varphi_n^4\}^{-1}\right). \tag{A.20}$$

Therefore, by combining (A.19) with (A.20), when T is large enough, we have

$$\begin{aligned}&\Pr\left(|\mathbf{B}_1(t_l^*, \boldsymbol{s}_l^*) - E(\mathbf{B}_1(t_l^*, \boldsymbol{s}_l^*))| > 2a(n,m)T\right) = O\left(\{a(n,m)T\varphi_n^4\}^{-1}\right)\\&+ O\left(n^{-T^{1/2}/65}\right) + O\left(n\exp\left(-mh_2^2\left(C_{\min}T^{1/2}-1\right)\right)\right)\\&+ O\left(n\exp\left\{-\log(n)\frac{C_1T^{1/2}}{10C_K}\right\}\right).\end{aligned} \tag{A.21}$$

By (A.15) and (A.21), it can be shown that, when T is large enough,

$$\begin{aligned}&\Pr\left(\sup_{(t,s)\in[0,1]\times\Omega}|\mathbf{B}_1(t,\boldsymbol{s}) - E(\mathbf{B}_1(t,\boldsymbol{s}))| > (2+4C_3)Ta(n,m)\right)\\&=O\left(\{a(n,m)^4h_1^3T\varphi_n^4\}^{-1}\right) + O\left(a(n,m)^{-3}h_1^{-3}n^{-T^{1/2}/65}\right)\\&+ O\left(a(n,m)^{-3}h_1^{-3}n\exp\{-mh_2^2\left(C_{\min}T^{1/2}-1\right)\}\right)\\&+ O\left(a(n,m)^{-3}h_1^{-3}n\exp\left\{-\log(n)\frac{C_1T^{1/2}}{20C_K}\right\}\right) = o(1).\end{aligned} \tag{A.22}$$

Note that $E(\mathbf{B}_1(t,\boldsymbol{s})) = 0$. So, by (A.22), we have $\mathbf{B}_1(t,\boldsymbol{s}) = O_p(a(n,m))$, which is uniformly true for all $(t,\boldsymbol{s}) \in [0,1]\times\Omega$. The vector of the remaining elements of $\mathbf{B}(t,\boldsymbol{s})$ can be proved in a similarly way to be of the order $\boldsymbol{H}\mathbf{1}O_p(a(n,m))$, where $\boldsymbol{H} = \text{diag}\{h_1,h_2,h_2\}$ and $\mathbf{1} = (1,1,1)^T$. Thus, we have

$$\mathbf{B}(t,\boldsymbol{s}) = \begin{pmatrix} O_p(a(n,m)) \\ \boldsymbol{H}\mathbf{1}O_p(a(n,m)) \end{pmatrix},$$

which are uniformly true for all $(t,\boldsymbol{s}) \in [0,1]\times\Omega$.

Next, we will study the properties of $\mathbf{A}(t,\boldsymbol{s})$. To this end, let $b(n,m) = h_1^2 + h_2^2 + \{\log(n)^2/(nh_1^2)\}^{1/2}$, and $b^*(n,m) = b(n,m) + \{\log(m)/(mh_1^2)\}^{1/2}$. Then, it can be shown by similar arguments to those for deriving (A.12)-(A.22) that

$$\mathbf{A}(t,\boldsymbol{s}) = \begin{pmatrix} a + O_P(b^*(n,m)) & \mathbf{1}^T\boldsymbol{H}O_p(b^*(n,m)) \\ \boldsymbol{H}\mathbf{1}O_p(b^*(n,m)) & \boldsymbol{C}(1+O_p(b^*(n,m))) \end{pmatrix},$$

where $a \in [C_{\min}, C_{\max}]$, all elements of the 3×3 matrix $\boldsymbol{C}$ are in the same order of the corresponding elements of $\boldsymbol{H}^2\boldsymbol{\mu}_2(K)$, and $\boldsymbol{\mu}_2(K)$ is defined in the proof of Theorem 1. By combining the above results, we have

$$\Pi_2 = \boldsymbol{e}_1^T\mathbf{A}(t,\boldsymbol{s})^{-1}\mathbf{B}(t,\boldsymbol{s}) = O_p(a(n,m)), \tag{A.23}$$

which is uniformly true for all $(t,\boldsymbol{s}) \in [0,1]\times\Omega$. For Π_3 defined in (A.2), in a similar way that we study the property of $\mathbf{B}(t,\boldsymbol{s})$, it can be checked that

$$\Pi_3 = O_p\left(h_1^2 + h_2^2\right), \tag{A.24}$$

which is uniformly true for all $(t,\boldsymbol{s}) \in [0,1]\times\Omega$. By combining the results in (A.1), (A.2), (A.23) and (A.24), the result (12) in Theorem 2 has been proved.

PROOF OF THEOREM 3

First, we consider the convergence property of $\hat{\sigma}^2(t,\boldsymbol{s})$. Since

$$\hat{\varepsilon}_{ij} - \varepsilon_{ij} = \left(y_{ij} - \hat{\lambda}(t_i,\boldsymbol{s}_{ij})\right) - \left(y_{ij} - \lambda(t_i,\boldsymbol{s}_{ij})\right) = \lambda(t_i,\boldsymbol{s}_{ij}) - \hat{\lambda}(t_i,\boldsymbol{s}_{ij}),$$

we know from Theorem 2 that $\hat{\varepsilon}_{ij} - \varepsilon_{ij}$ is bounded by a term of the order $O_p(b(n,m))$ uniformly for all i and j. Then we have

$$\begin{aligned}\hat{\sigma}^2(t,\boldsymbol{s}) =&\frac{\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)\hat{\varepsilon}_{ij}^2}{\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)} = \frac{\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)\varepsilon_{ij}^2}{\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)}\\&+ O_p(b(n,m)) =: \Pi_5/\Pi_4 + O_p(b(n,m)),\end{aligned} \tag{A.25}$$

where $\Pi_4 = (nmh_3h_4^2f(\mathbf{s}))^{-1}\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)$ and $\Pi_5 = (nmh_3h_4^2f(\mathbf{s}))^{-1}\sum_{i=1}^n\sum_{j=1}^{m_i} w_1(i,j)\varepsilon_{ij}^2$. Next, we will show that $\Pi_5/\Pi_4 = \sigma^2(t,\mathbf{s}) + O_p(\widetilde{\nu}(n,m))$, where $\widetilde{\nu}(n,m) = h_3^2 + h_4^2 + \{nh_3\}^{-1/2}$. By a simple calculation, we have

$$\begin{aligned} E\left(\Pi_5\right) &= \frac{1}{nh_3f(\mathbf{s})}\sum_{i=1}^n \frac{m_i}{m}K_1\left((t_i-t)/h_3\right)\left[\sigma^2(t_i,\mathbf{s})f(\mathbf{s}) + O(h_4^2)\right] \\ &= \sigma^2(t,\mathbf{s})\frac{1}{nh_3}\sum_{i=1}^n \frac{m_i}{m}K_1\left((t_i-t)/h_3\right) + O(\widetilde{\nu}(n,m)). \end{aligned} \tag{A.26}$$

By similar arguments to those in (A.6)-(A.9), it can be checked that $\mathrm{Var}(\Pi_5) = O(1/(nh_3))$. By combining this result with that in (A.26), we have

$$\Pi_5 = \sigma^2(t,\mathbf{s})\frac{1}{nh_3}\sum_{i=1}^n \frac{m_i}{m}K_1\left((t_i-t)/h_3\right) + O_p(\widetilde{\nu}(n,m)). \tag{A.27}$$

Similarly, it can be shown that

$$\Pi_4 = \frac{1}{nh_3}\sum_{i=1}^n \frac{m_i}{m}K_1\left((t_i-t)h_3\right) + O_p(\widetilde{\nu}(n,m) + \{mh_4^2\}^{-1/2}), \tag{A.28}$$

It follows from (A.25), (A.27) and (A.28) that the result in (13) of the paper is true.

Second, for any $\varpi \geq 0$, $\mathbf{s},\mathbf{s}' \in \Omega$, and $t,t' \in [0,1]$ such that $n(t'-t) = \varpi + o(1)$, we consider the convergence property of $\widehat{V}(t,t';\mathbf{s},\mathbf{s}')$. To this end, we first decompose $\widehat{V}(t,t';\mathbf{s},\mathbf{s}') - V(t,t';\mathbf{s},\mathbf{s}')$ into three parts. Define

$$\begin{aligned} \widetilde{V}(t,t';\mathbf{s},\mathbf{s}') =& \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\varepsilon_{ij}\varepsilon_{kl}}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)}, \quad \text{and} \\ V^*(t,t';\mathbf{s},\mathbf{s}') =& \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)V(t_i,t_k;\mathbf{s}_{ij},\mathbf{s}_{kl})}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)}. \end{aligned}$$

Then, it is straightforward that

$$\begin{aligned} \widehat{V}(t,t';\mathbf{s},\mathbf{s}') - V(t,t';\mathbf{s},\mathbf{s}') =& \{\widetilde{V}(t,t';\mathbf{s},\mathbf{s}') - V^*(t,t';\mathbf{s},\mathbf{s}')\} + \{V^*(t,t';\mathbf{s},\mathbf{s}') - V(t,t';\mathbf{s},\mathbf{s}')\} \\ &+ \{\widehat{V}(t,t';\mathbf{s},\mathbf{s}') - \widetilde{V}(t,t';\mathbf{s},\mathbf{s}')\} =: \Lambda_1/\Lambda_2 + \Lambda_3/\Lambda_2 + \Lambda_4/\Lambda_2, \end{aligned} \tag{A.29}$$

where

$$\begin{aligned} \Lambda_1 =& \sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\left(\varepsilon_{ij}\varepsilon_{kl} - V(t_i,t_k;\mathbf{s}_{ij},\mathbf{s}_{kl})\right), \Lambda_2 = \sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l), \\ \Lambda_3 =& \sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\left(V(t_i,t_k;\mathbf{s}_{ij},\mathbf{s}_{kl}) - V(t,t';\mathbf{s},\mathbf{s}')\right), \Lambda_4 = \sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\left(\widehat{\varepsilon}_{ij}\widehat{\varepsilon}_{kl} - \varepsilon_{ij}\varepsilon_{kl}\right). \end{aligned}$$

Next, we will show that each of Λ_1/Λ_2, Λ_3/Λ_2 and Λ_4/Λ_2 can be bounded by a term of the order $O_p(\widetilde{\nu}(n,m)+b(n,m))$, where $\widetilde{\nu}(n,m) = h_3^2+h_4^2+\{nh_3\}^{-1/2}$ and $b(n,m) = h_1^2+h_2^2+\{\log(n)^2/(nh_1^2)\}^{1/2}$. To this end, let us first consider Λ_1. From its definition, it can be checked that $E\left(\Lambda_1|\mathcal{S}_\sigma\right) = 0$, where $\mathcal{S}_\sigma$ is the σ-algebra generated by $\mathcal{S} = \{\mathbf{s}_{11},\ldots,\mathbf{s}_{nm}\}$. It follows from this that

$$E\left(\Lambda_1\right) = E\left[E\left(\Lambda_1|\mathcal{S}_\sigma\right)\right] = 0. \tag{A.30}$$

Denote $R(i,j,k,l;i',j',k',l') = E\left\{w_2(i,j,k,l)w_2(i',j',k',l')\right\}|\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl},\varepsilon_{i'j'}\varepsilon_{k'l'})|$. To calculate the variance of Λ_1, since $E\left(\Lambda_1|\mathcal{S}_\sigma\right) = 0$ and $\mathrm{Var}(\Lambda_1) = \mathrm{Var}\left(E\left(\Lambda_1|\mathcal{S}_\sigma\right)\right) + E\left[\mathrm{Var}\left(\Lambda_1|\mathcal{S}_\sigma\right)\right]$, we have

$$\begin{aligned} \mathrm{Var}(\Lambda_1) =& \sum_{i,j}\sum_{(k,l)\neq(i,j)}\sum_{i',j'}\sum_{(k',l')\neq(i',j')} E\left\{w_2(i,j,k,l)w_2(i',j',k',l')\right\}\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl},\varepsilon_{i'j'}\varepsilon_{k'l'}) \\ \leq& \sum_{i,j}\sum_{k,l}\sum_{i',j'}\sum_{k',l'} R(i,j,k,l;i',j',k',l') \leq 2\sum_{i,j}\sum_{k,l}\sum_{i'\geq i,j'}\sum_{k',l'} R(i,j,k,l;i',j',k',l') \\ =& O(1)\times\sum_{i,j}\sum_{k\geq i,l}\sum_{i'\geq i,j'}\sum_{k'\geq i',l'} R(i,j,k,l;i',j',k',l') = O(1)\times(\Lambda_{1,1}+\Lambda_{1,2}+\Lambda_{1,3}), \end{aligned} \tag{A.31}$$

where

$$\Lambda_{1,1} = \sum_{i,j} \sum_{i'\geq i,j'} \sum_{k'\geq i',l'} \sum_{i\leq k\leq i',l} R(i,j,k,l;i',j',k',l'),$$
$$\Lambda_{1,2} = \sum_{i,j} \sum_{i'\geq i,j'} \sum_{k'\geq i',l'} \sum_{i'\leq k\leq k',l} R(i,j,k,l;i',j',k',l'), \quad \text{(A.32)}$$
$$\Lambda_{1,3} = \sum_{i,j} \sum_{i'\geq i,j'} \sum_{k'\geq i',l'} \sum_{k\geq k',l} R(i,j,k,l;i',j',k',l').$$

Next, we will find the upper bounds for $\Lambda_{1,1}$, $\Lambda_{1,2}$ and $\Lambda_{1,3}$, respectively. To this end, we first consider $\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl}, \varepsilon_{i'j'}\varepsilon_{k'l'})$, for any i, j, k, l, i', j', k' and l' such that $i \leq k$, $i' \leq k'$ and $i \leq i'$. Note that, if $i \leq k \leq i'$, by the Davydov's inequality, we have

$$|\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl}, \varepsilon_{i'j'}\varepsilon_{k'l'})| \leq C_4 \exp(-C_5|i'-k|), \quad \text{(A.33)}$$

where $C_4 = 12C_\varepsilon^{4/\delta}C_0^{(\delta-4)/\delta}$ and $C_5 = C_1(\delta-4)/\delta$. If $i' < k \leq k'$, then it can be shown that

$$\begin{aligned}|\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl}, \varepsilon_{i'j'}\varepsilon_{k'l'})| \leq& |\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{i'j'}, \varepsilon_{kl}\varepsilon_{k'l'})| + |E(\varepsilon_{ij}\varepsilon_{kl})E(\varepsilon_{i'j'}\varepsilon_{k'l'})| + |E(\varepsilon_{ij}\varepsilon_{i'j'})E(\varepsilon_{kl}\varepsilon_{k'l'})| \\ \leq& C_4\left\{\exp(-C_5|i'-k|) + \exp(-C_5|i'-k'|) + \exp(-C_5|i'-i|)\right\}.\end{aligned} \quad \text{(A.34)}$$

If $k' < k$, it is clear that

$$|\mathrm{Cov}(\varepsilon_{ij}\varepsilon_{kl}, \varepsilon_{i'j'}\varepsilon_{k'l'})| \leq 2C_4. \quad \text{(A.35)}$$

Denote $\Delta_n(t,t') = \left(|t-t'| - 1/n, |t-t'| + 1/n\right)$. For the quantity $\Lambda_{1,1}$, it can be shown from (A.33) that

$$\begin{aligned}\Lambda_{1,1} =& O(1) \times m^4 h_4^8 \sum_i \sum_{i'\geq i} \sum_{k'\geq i'} \sum_{i\leq k\leq i'} K_1(\frac{t_i - t}{h_3})K_1(\frac{t_k - t'}{h_3})K_1(\frac{t_{i'} - t}{h_3})K_1(\frac{t_{k'} - t'}{h_3}) \\ &\times \mathrm{I}\left(|t_i - t_k| \in \Delta_n(t,t')\right) \mathrm{I}\left(|t_{i'} - t_{k'}| \in \Delta_n(t,t')\right) \exp\left(-C_5|i'-k|\right).\end{aligned} \quad \text{(A.36)}$$

Note that, for every integer number k, the number of different i's such that $\mathrm{I}\left(|t_i - t_k| \in \Delta_n(t,t')\right) = 1$ cannot exceed 3. Meanwhile, given the value of i', we have at most 3 different k''s such that $\mathrm{I}\left(|t_{i'} - t_{k'}| \in \Delta_n(t,t')\right) = 1$. Thus, it follows from (A.36) that

$$\begin{aligned}\Lambda_{1,1} =& O(1) \times 9C_K^3 m^4 h_4^8 \sum_{k=1}^n \sum_{i'=1}^n K_1(\frac{t_k - t'}{h_3}) \exp\left(-C_5|i'-k|\right) \\ =& O(m^4 h_4^8) \times \sum_{k=1}^n K_1(\frac{t_k - t'}{h_3}) \sum_{\tau=0}^{\infty} \exp(-C_5\tau) \\ =& O(nm^4 h_3 h_4^8) \times \frac{1}{nh_3} \sum_{k=1}^n K_1(\frac{t_k - t'}{h_3}),\end{aligned} \quad \text{(A.37)}$$

where $C_K = \sup_{x\in\mathbb{R}}\{K_1(x), K_2(x)\}$. *When t' is fixed, it has been well studied that $(nh_3)^{-1}\sum_{k=1}^n K_1\left((t_k - t')/h_3\right) = 1 + O\left(1/(nh_3)\right)$. However, in the setup of Theorem 3 of the paper, t' may change with the number of observation times n. As a result, in order to obtain the upper bound of $\Lambda_{1,1}$, we have to calculate the upper bound of* $\sup_{u\in[0,1]}(nh_3)^{-1}\sum_{k=1}^n K_1\left((t_k - u)/h_3\right)$. Define $\Delta_k = (t_{k-1}, t_k]$, for $k = 1, \ldots, n$, where $t_0 = 0$ and $\{t_k = k/n, k = 1, \ldots, n\}$. Then, for any $u \in [0,1]$, it can be shown that

$$\begin{aligned}\frac{1}{nh_3}\sum_{k=1}^n K_1(\frac{t_k - u}{h_3}) =& \frac{1}{h_3}\sum_{k=1}^n \int K_1(\frac{t_k - u}{h_3})\mathrm{I}(x \in \Delta_k)dx = \frac{1}{h_3}\sum_{k=1}^n \int K_1(\frac{x-u}{h_3})\mathrm{I}(x \in \Delta_k)dx \\ &+ \frac{1}{h_3}\sum_{k=1}^n \int \left\{K_1(\frac{t_k - u}{h_3}) - K_1(\frac{x-u}{h_3})\right\}\mathrm{I}(x \in \Delta_k)dx \\ \leq& \int K_1(z)dz + \frac{1}{h_3}\sum_{k=1}^n \int \left\{K_1(\frac{t_k - u}{h_3}) - K_1(\frac{x-u}{h_3})\right\}\mathrm{I}(x \in \Delta_k)dx \\ =& 1 + \frac{1}{h_3}\sum_{k=1}^n \int \left\{K_1(\frac{t_k - u}{h_3}) - K_1(\frac{x-u}{h_3})\right\}\mathrm{I}(x \in \Delta_k)dx.\end{aligned} \quad \text{(A.38)}$$

Note that when $x \in \Delta_k$, $\left|K_1\left((t_k-u)/h_3\right)-K_1\left((x-u)/h_3\right)\right| \le L_K|x-t_k|h_3^{-1}\mathrm{I}(|t_k-u|\le 2L_1h_3)$, where L_K is the Lipschitz constant of the kernel functions and $[-L_1, L_1]$ is the finite support of $K_1(x)$. It follows that

$$\begin{aligned}
&\left|\frac{1}{h_3}\sum_{k=1}^{n}\int\left\{K_1(\frac{t_k-u}{h_3})-K_1(\frac{x-u}{h_3})\right\}\mathrm{I}(x\in\Delta_k)dx\right|\\
\le&\frac{1}{h_3}\sum_{k=1}^{n}\int\left|K_1(\frac{t_k-u}{h_3})-K_1(\frac{x-u}{h_3})\right|\mathrm{I}(x\in\Delta_k)dx\\
\le&\frac{1}{h_3}\sum_{k=1}^{n}\int L_K|x-t_k|h_3^{-1}\mathrm{I}(|t_k-u|\le 2L_1h_3)\mathrm{I}(x\in\Delta_k)dx\\
\le&L_K(nh_3^2)^{-1}\sum_{k=1}^{n}\int\mathrm{I}(|t_k-u|\le 2L_1h_3)\mathrm{I}(x\in\Delta_k)dx\\
\le&L_K(nh_3^2)^{-1}(4L_1h_3+2/n).
\end{aligned} \tag{A.39}$$

combining the results in (A.39) and (A.38), it immediately follows that $(nh_3)^{-1}\sum_{k=1}^{n}K_1\left((t_k-u)/h_3\right)\le 1+L_K(nh_3^2)^{-1}(4L_1h_3+2/n)$, for any $u\in[0,1]$. Thus,

$$\sup_{u\in[0,1]}(nh_3)^{-1}\sum_{k=1}^{n}K_1\left((t_k-u)/h_3\right)\le 1+L_K(nh_3^2)^{-1}(4L_1h_3+2/n)=1+O(\{nh_3\}^{-1}). \tag{A.40}$$

Based on the results in (A.37) and (A.40), we have

$$\begin{aligned}
\Lambda_{1,1}=&O(nm^4h_3h_4^8)\times(nh_3)^{-1}\sum_{k=1}^{n}K_1\left((t_k-t')/h_3\right)\\
=&O(nm^4h_3h_4^8)\times\sup_{u\in[0,1]}(nh_3)^{-1}\sum_{k=1}^{n}K_1\left((t_k-u)/h_3\right)=O(nm^4h_3h_4^8).
\end{aligned} \tag{A.41}$$

To calculate the second part $\Lambda_{1,2}$ in (A.32), note that the number of different k's such that $i'<k\le k'$ is no more than $|k'-i'|$ and $|k'-i'|\le\tau_n+1$, where τ_n is the closest positive integer less than or equal to $n|t-t'|$. By using the result in (A.34), we have

$$\begin{aligned}
\Lambda_{1,2}=&O(1)\times m^4h_4^8\sum_i\sum_{i'\ge i}\sum_{k'\ge i'}\sum_{i'<k\le k'}K_1(\frac{t_i-t}{h_3})K_1(\frac{t_k-t'}{h_3})K_1(\frac{t_{i'}-t}{h_3})K_1(\frac{t_{k'}-t'}{h_3})\mathrm{I}\left(|t_i-t_k|\in\Delta_n(t,t')\right)\\
&\times\mathrm{I}\left(|t_{i'}-t_{k'}|\in\Delta_n(t,t')\right)\left\{\exp\left(-C_5|i'-k|\right)+\exp\left(-C_5|i'-k'|\right)+\exp\left(-C_5|i'-i|\right)\right\}\\
=&O(nh_3m^4h_4^8)+O(m^4h_4^8)\times\sum_i(\tau_n+1)K_1(\frac{t_i-t}{h_3})\exp\left(-C_5(\tau_n-1)\right)\\
&+O(m^4h_4^8)\times\sum_i\sum_{i'}K_1(\frac{t_i-t}{h_3})K_1(\frac{t_{i'}-t}{h_3})\exp\left(-C_5|i'-i|\right)\\
=&O(nh_3m^4h_4^8)+O(nh_3m^4h_4^8)\times\sup_{u\in[0,1]}\frac{1}{nh_3}\sum_{i=1}^{n}K_1(\frac{t_i-u}{h_3})\sum_{\tau=0}^{\infty}\exp\left(-C_5\tau\right)=O(nh_3m^4h_4^8).
\end{aligned} \tag{A.42}$$

Note that $k-i\le k'-i'+2$ if both $|t_k-t_i|$ and $|t_{k'}-t_{i'}|$ are in $\Delta_n(t,t')$, $i\le k$, and $i'\le k'$. It follows that $k\le k'+2-(i'-i)$. When $i'\ge i$, the number of different k's satisfying $k>k'$ and $k\le k'+2-(i'-i)$ is less than or equal to 2. From the definition of $\Lambda_{1,3}$ and the result in (A.35), it can be easily checked that

$$\Lambda_{1,3}=O(nh_3m^4h_4^8). \tag{A.43}$$

From the results in (A.41)-(A.43), we have $\mathrm{Var}(\Lambda_1)=O(nm^4h_3h_4^8)$. It follows from this result and (A.30) that

$$\Lambda_1=O_p(nm^2h_3h_4^4\widetilde{v}(n,m)). \tag{A.44}$$

For Λ_2, by using the similar arguments in (A.30)-(A.44), we can obtain the result that

$$\begin{aligned}
\Lambda_2=&\sum_{i=1}^{n}\sum_{k=1}^{n}K_1(\frac{t_i-t}{h_3})K_1(\frac{t_k-t'}{h_3})\mathrm{I}\left(|t_i-t_k|\in\Delta_n(t,t')\right)\\
&\times m_im_kf(\mathbf{s})f(\mathbf{s}')h_4^4+O_p(nm^2h_3h_4^4\widetilde{v}^*(n,m)),
\end{aligned} \tag{A.45}$$

where $\widetilde{\nu}^*(n,m) = \widetilde{\nu}(n,m) + \{mh_4^2\}^{-1/2}$. Since $n(t-t') = \varpi + o(1)$, when n is large enough, for arbitrary $1 \le i \le n$, we can find at least one integer $1 \le k \le n$ such that $\mathrm{I}\left(|t_i - t_k| \in \Delta_n(t,t')\right) = 1$. Moreover, it can be easily checked that $\Big| K_1\left((t_i - t)/h_3\right) - K_1\left((t_k - t')/h_3\right) \Big| \le L_K (nh_3)^{-1}$ when $\mathrm{I}\left(|t_i - t_k| \in \Delta_n(t,t')\right) = 1$. So, we have

$$\begin{aligned}
&(nh_3)^{-1} \sum_{i=1}^{n} \sum_{k=1}^{n} K_1(\frac{t_i - t}{h_3}) K_1(\frac{t_k - t'}{h_3}) \mathrm{I}\left(|t_i - t_k| \in \Delta_n(t,t')\right) \\
\ge &(nh_3)^{-1} \sum_{i=1}^{n} K_1^2(\frac{t_i - t}{h_3}) - L_K (nh_3)^{-2} \sum_{i=1}^{n} K_1(\frac{t_i - t}{h_3}) \\
\ge &(nh_3)^{-1} \inf_{u \in [0,1]} \sum_{i=1}^{n} K_1^2(\frac{t_i - u}{h_3}) - L_K (nh_3)^{-2} \sup_{u \in [0,1]} \sum_{i=1}^{n} K_1(\frac{t_i - u}{h_3}) \\
= &\frac{1}{2}\mu(K_1^2) + O(\frac{1}{nh_3}),
\end{aligned} \tag{A.46}$$

where $\mu(K_1^2) = \int K_1^2(x)dx$. It follows from (A.44)-(A.46) that

$$\Lambda_1/\Lambda_2 = O_p(\widetilde{\nu}(n,m)). \tag{A.47}$$

For Λ_3, note that $V(t,t';s,s')$ is twice continuously differentiable. By the Taylor's expansion, the following result is true:

$$\Lambda_3 = O_p(nm^2 h_3 h_4^4 \widetilde{\nu}(n,m)). \tag{A.48}$$

From (A.45), (A.46) and (A.48), we have

$$\Lambda_3/\Lambda_2 = O_p(\widetilde{\nu}(n,m)). \tag{A.49}$$

So far, we have shown that $\Lambda_1/\Lambda_2 + \Lambda_3/\Lambda_2 = O_p(\widetilde{\nu}(n,m))$. To prove that the result in (14) of the paper is true, it suffices to show that $\Lambda_4/\Lambda_2 = O_p(\widetilde{\nu}(n,m))$. From the definition of Λ_2 and Λ_4 and the result that $\widehat{\varepsilon}_{ij} - \varepsilon_{ij}$ is bounded by a term of the order $O_p(b(n,m))$ uniformly for all i and j (see the arguments at the beginning of the proof), we have

$$\begin{aligned}
|\Lambda_4/\Lambda_2| \le &\frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\left|\widehat{\varepsilon}_{ij}\widehat{\varepsilon}_{kl} - \varepsilon_{ij}\varepsilon_{kl}\right|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} \le \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)\left|\widehat{\varepsilon}_{ij} - \varepsilon_{ij}\right|\left|\widehat{\varepsilon}_{kl} - \varepsilon_{kl}\right|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} \\
&+ \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{kl}|\left|\widehat{\varepsilon}_{ij} - \varepsilon_{ij}\right|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} + \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{ij}|\left|\widehat{\varepsilon}_{kl} - \varepsilon_{kl}\right|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} \\
= &O_p(b(n,m)^2) + O_p(b(n,m)) \times \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{kl}|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} \\
&+ O_p(b(n,m)) \times \frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{ij}|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)}.
\end{aligned} \tag{A.50}$$

Similar to the arguments in (A.31)-(A.47), it can be shown that

$$\begin{aligned}
\frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{kl}|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} &= O_p(1), \quad \text{and} \\
\frac{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)|\varepsilon_{ij}|}{\sum_{i,j}\sum_{(k,l)\neq(i,j)} w_2(i,j,k,l)} &= O_p(1).
\end{aligned} \tag{A.51}$$

The results in (A.50) and (A.51) imply that

$$\Lambda_4/\Lambda_2 = O_p(b(n,m)). \tag{A.52}$$

By combining the results in (A.47), (A.49) and (A.52), the result in (14) of the paper has been proved.

□