Supplemental file for the paper titled "Nonparametric Profile Monitoring By Mixed Effects Modeling"

Appendix: Technical details

Throughout the appendix, we use the following additional notations:

$$\alpha_{t,h,\lambda}(s) = \frac{\nu^2(s)}{a_{0,t,\lambda}\Gamma_2(s)} \sum_{k=1}^{t} (1-\lambda)^{t-k} \sum_{j=1}^{n_k} \frac{K_h(x_{kj}-s)}{\nu^2(x_{kj})}[\varepsilon_{kj} + f_k(x_{kj})],$$

$$\beta_{t,h,\lambda}(s) = \frac{g_1''(s)\nu^2(s)}{2a_{0,t,\lambda}\Gamma_2(s)} \sum_{k=1}^{t} (1-\lambda)^{t-k} \sum_{j=1}^{n_k} (x_{kj}-s)^2 \frac{K_h(x_{kj}-s)}{\nu^2(x_{kj})},$$

$$\phi_i(s) = \frac{1}{a_{0,t,\lambda}} \sum_{k=1}^{t} (1-\lambda)^{t-k} \sum_{j=1}^{n_k} (x_{kj}-s)^i \frac{K_h(x_{kj}-s)}{\nu^2(x_{kj})}[\varepsilon_{kj} + f_k(x_{kj})], \quad i = 0, 1,$$

$$\phi_{i+2}(s) = \frac{1}{a_{\tau,t,\lambda}} \sum_{k=\tau+1}^{t} (1-\lambda)^{t-k} \sum_{j=1}^{n_k} (x_{kj}-s)^i \frac{K_h(x_{kj}-s)}{\nu^2(x_{kj})} g_1(x_{kj}), \quad i = 0, 1,$$

$$e_{t_0,t_1,\lambda} = \sum_{k=t_0+1}^{t_1} (1-\lambda)^{4(t-k)} n_k^2, \quad n^* = m/(\sum_{i=1}^{m} n_i^{-1}), \quad \eta_2 = \int [K(u)]^2 du.$$

Appendix A: Regularity Conditions Used In Section 2

(C1) Density functions Γ_1 and Γ_2 are Lipschitz continuous and bounded away from zero on [0,1].

(C2) $g_0(\cdot)$ and $g_1(\cdot)$ have continuous second order derivatives on [0,1].

(C3) The kernel function $K(u)$ is bounded and symmetric about 0 on [-1,1]. Furthermore, $u^3K(u)$ and $u^3K'(u)$ are both bounded, and $\int_{-1}^{1} u^4 K(u)du < \infty$.

(C4) $E(|\varepsilon_{11}|^4) < \infty$.

(C5) The covariance function $\gamma(s,t)$ has continuous second order derivatives for both s and t.

(C6) n_i, for $i = 1, \ldots, m$, are of the same order, say $n_i \sim n$.

(C7) $\frac{a_{\tau,t,\lambda}}{a_{0,t,\lambda}} - 1 = o(\min\{h^2, c_{0,t,\lambda}^{-\frac{1}{2}}\})$.

(C8)-(I) n_i, m, h satisfy the conditions that $n_i \to \infty$, $m \to \infty$, $h \to 0$, $n_i h^2 \to \infty$, $mn_i h^3 \to \infty$.

(C8)-(II) n_0, h and $c_{0,t,\lambda}$ satisfy the conditions that $n_0 \to \infty$, $h \to 0$, $n_0 h^{\frac{3}{2}} \to \infty$, $c_{0,t,\lambda} h^{\frac{3}{2}} \to \infty$, $c_{0,t,\lambda} h^8 \to 0$ and $e_{0,t,\lambda}/(hb_{0,t,\lambda}^2) \to 0$.

(C8)-(III) h and $c_{0,t,\lambda}$ satisfy the conditions that $h \to 0$, $c_{0,t,\lambda} h^3 \to \infty$ and $c_{0,t,\lambda} h^8 \to 0$.

(C8)-(IV) n_0, h and $c_{0,t,\lambda}$ satisfy the conditions that $n_0 \to \infty$, $h \to 0$, $n_0 h^{\frac{3}{2}} \to \infty$, $c_{0,t,\lambda} h^3 \to \infty$, $c_{0,t,\lambda} n h^5 \to 0$ and $e_{0,t,\lambda}/(hb_{0,t,\lambda}^2) \to 0$.

(C9) $\mathbf{D}_{(0)}$ is assumed to be a positive definite matrix.

(C10) For some $w > 2$, $E(|\varepsilon_{11} + f_1(x_{11})|^w) < \infty$ and $t\lambda \to \infty$.

It is noted that conditions (C1)-(C4) are standard in nonparametric regression. (C5) assures the smoothness of the random-effects $f_i(\cdot)$. (C6) is for technical convenience and is often satisfied in practice. (C7) is easy to satisfy if t is large enough. (C8)-I to (C8)-IV are the bandwidth conditions used in Proposition 1, Theorem 1-(i), Theorem 1-(ii) and Theorem 2, respectively. (C9) is a mild condition for Proposition 1. (C10) is an extra condition for the proofs of Theorem 1-(ii) and Theorem 2-(iii).

Appendix B: Proofs

Proof of Proposition 1

(i) This result is a direct conclusion of Theorem 1 in Wu and Zhang (2002), except that certain conditions have been changed. More specifically, expressions (7) and (8) of Condition A in Wu and Zhang (2002) have been relaxed to Conditions (C8)-I and (C9) here. The proof of this conclusion is similar to that of result (ii), and thus it is omitted here.

(ii) To investigate $\widehat{\gamma}(\cdot,\cdot)$, we first need to study the asymptotic behavior of $\widehat{g}'(\cdot)$. Let

$$s_{i,r}(x) = \frac{1}{n_i h^r}\sum_{j=1}^{n_i} K_h(x_{ij}-x)(x_{ij}-x)^r, \quad r = 0, 1, 2,$$

$$\mathbf{A} = [\sigma^2_{(0)}]^{-1}\mathbf{D}_{(0)}, \quad \mathbf{G}_i = n_i \begin{pmatrix} s_{i,0} & hs_{i,1} \\ hs_{i1} & h^2 s_{i,2} \end{pmatrix}.$$

Similar to Proposition 1 in Wu and Zhang (2002), we have

$$\widehat{g}'(x) = \boldsymbol{e}_2^T\{\sum_{i=1}^m (\mathbf{I}+\mathbf{G}_i\mathbf{A})^{-1}\mathbf{G}_i\}^{-1}\sum_{i=1}^m(\mathbf{I}+\mathbf{G}_i\mathbf{A})^{-1}\boldsymbol{Z}_i^T\mathbf{K}_i\boldsymbol{y}_i, \tag{A.1}$$

where $\boldsymbol{e}_2 = (0,1)^T$. Note that $s_{i,r}(x) - s_r\Gamma_2(x) = O_p(n_i h)^{-\frac{1}{2}} + O(h)$, where $s_r = \int K(u)u^r du$ with $s_0 = 1, s_1 = 0$, and $s_2 = \eta_1$. By Condition (C9), we have

$$\mathbf{G}_i^{-1}\mathbf{A}^{-1} = \frac{1}{n_i\Gamma_2(x)(h^2 s_{i2}s_{i0} - h^2 s_{i1})}\begin{pmatrix} O_p(h^2 + n_i^{-\frac{1}{2}}h^{\frac{1}{2}}) & O_p(h^2 + n_i^{-\frac{1}{2}}h^{\frac{1}{2}}) \\ O_p(1) & O_p(1)\end{pmatrix}. \tag{A.2}$$

By noting that $h^2 s_{i2}s_{i0} - h^2 s_{i1} = O(h^2)$ and by Condition (C8)-I that $n_i h^2 \to \infty$, we have $\mathbf{G}_i^{-1}\mathbf{A}^{-1} = o_p(1)$. Consequently, $\frac{1}{m}\sum_{i=1}^m \mathbf{G}_i^{-1}\mathbf{A}^{-1} = o_p(1)$. By using certain matrix manipulations similar to those in Wu and Zhang (2002), (A.1) leads to

$$\widehat{g}'(x) = g'(x) + \frac{1}{m}\sum_{i=1}^m u_i(x)(1+o_p(1)),$$

where

$$\begin{aligned} u_i(x) &= \sum_{j=1}^{n_i}\boldsymbol{e}_2^T\mathbf{G}_i^{-1}\begin{pmatrix} 1 \\ x_{ij}-x\end{pmatrix} K_h(x_{ij}-x)[\varepsilon_{ij} + f_i(x_{ij}) + \frac{1}{2}g''(x)(x_{ij}-x)^2] \\ &= \sum_{j=1}^{n_i}\left[\frac{K_h(x_{ij}-x)(x_{ij}-x)}{\eta_1\Gamma_2(x)n_i h^2} - \frac{s_{i,1}}{n_i h\eta_1\Gamma_2(x)}\right][\varepsilon_{ij} + f_i(x_{ij}) + \frac{1}{2}g''(x)(x_{ij}-x)^2]. \end{aligned}$$

It is straightforward to show that

$$E[u_i(x)|\boldsymbol{X}] = O_p(h^2), \quad \mathrm{Var}[u_i(x)|\boldsymbol{X}] = [\gamma(x,x) + \frac{\nu^2(x)\eta_2}{n_i h^3}](1+o(1)).$$

Thus, we have

$$\widehat{g}'(x) = g'(x)[1 + O_p(h^2) + O_p(m^{-\frac{1}{2}}) + O_p((mn^*h^3)^{-\frac{1}{2}})]. \tag{A.3}$$

Now, $\widehat{f}_i(x)$ can be rewritten as

$$\widehat{f}_i(x) = \boldsymbol{e}_1^T \mathbf{A}(\mathbf{I} + \mathbf{G}_i\mathbf{A})^{-1}\boldsymbol{Z}_i^T \mathbf{K}_i[\boldsymbol{y}_i - \boldsymbol{Z}_i(\widehat{\boldsymbol{\beta}} + \boldsymbol{\alpha}_i) + \boldsymbol{Z}_i\boldsymbol{\alpha}_i].$$

By using the fact that $\mathbf{G}_i^{-1}\mathbf{A}^{-1} = o_p(1)$, we have

$$\begin{aligned}
\boldsymbol{e}_1^T \mathbf{A}(\mathbf{I} + \mathbf{G}_i\mathbf{A})^{-1}\boldsymbol{Z}_i^T \mathbf{K}_i \boldsymbol{Z}_i\boldsymbol{\alpha}_i &= \boldsymbol{e}_1^T[\mathbf{I} + (\mathbf{A}\mathbf{G}_i)^{-1}]^{-1}\boldsymbol{\alpha}_i \\
&= \boldsymbol{e}_1^T[\mathbf{I} - (\mathbf{A}\mathbf{G}_i)^{-1} + (\mathbf{A}\mathbf{G}_i)^{-2} + \cdots]\boldsymbol{\alpha}_i \\
&= [\boldsymbol{e}_1^T\boldsymbol{\alpha}_i - \boldsymbol{e}_1^T(\mathbf{A}\mathbf{G}_i)^{-1}\boldsymbol{\alpha}_i](1 + o_p(1)) \\
&= f_i(x)[1 + O_p(n_i^{-1} + (n_ih)^{-\frac{3}{2}})], \qquad \text{(A.4)}
\end{aligned}$$

where the last equation is from (A.2).

Because $\mathbf{A}(\mathbf{I} + \mathbf{G}_i\mathbf{A})^{-1} = \mathbf{G}_i^{-1}(1 + o(1))$, by (A.4) and (C5), we have

$$\begin{aligned}
\widehat{f}_i(x) - f_i(x) =& \sum_{j=1}^{n_i} \frac{K_h(x_{ij} - x)}{n_i\Gamma_2(x)}\{\varepsilon_{ij} + [g(x) - \widehat{g}(x)] + [g'(x) - \widehat{g}'(x)](x_{ij} - x) \\
&+ \frac{1}{2}[g''(x) + f_i''(x)](x_{ij} - x)^2\}(1 + o(1)) + O_p[n_i^{-1} + (n_ih)^{-\frac{3}{2}}] \\
=& \sum_{j=1}^{n_i} \frac{K_h(x_{ij} - x)}{n_i\Gamma_2(x)}[\varepsilon_{ij} + \frac{1}{2}f_i''(x)(x_{ij} - x)^2](1 + o_p(1)) =: v_i(x).
\end{aligned}$$

By certain straightforward calculations, we have

$$\begin{aligned}
E[v_i(x)|\boldsymbol{X}] = O(h^2) + O_p(m^{-\frac{1}{2}}), &\quad \mathrm{Var}[v_i(x)|\boldsymbol{X}] = O_p[h^2 + (n_ih)^{-1} + (mn_ih^2)^{-1}], \\
E[v_i(s_k)v_i(s_l)|\boldsymbol{X}] = O_p[h^2 + (n_ih)^{-1}], &\quad \mathrm{Var}[v_i(s_k)v_i(s_l)|\boldsymbol{X}] = O_p[h^4 + (n_ih^3)^{-1}], \\
\mathrm{Var}[v_i(s_k)v_j(s_l)|\boldsymbol{X}] = O_p[h^4 + (n_ih)^{-2} + m^{-1}h^2], &\quad \text{for} \quad i \neq j.
\end{aligned}$$

It follows that

$$\begin{aligned}
\widehat{\gamma}(s_k, s_l) &= \frac{1}{m}\sum_{i=1}^{m}[f_i(s_k) + v_i(s_k)][f_i(s_l) + v_i(s_l)] \\
&= \gamma(s_k, s_l)\{1 + O_p[h^2 + (nh)^{-\frac{1}{2}} + m^{-\frac{1}{2}} + (mnh^3)^{-\frac{1}{2}}]\}.
\end{aligned}$$

(iii) The proof of this result is analogous to that of result (ii), and it is omitted here. $\square$

To prove the two theorems in Section 2, we first prove the following lemma.

Lemma 1 *For any $s \in [0,1]$,*

(i) Under conditions in Theorem 1, we have

$$\widehat{g}_{t,h,\lambda}(s) = \alpha_{t,h,\lambda}(s)(1 + o(h^{\frac{1}{2}}));$$

(ii) Under conditions in Theorem 2, we have

$$\widehat{g}_{t,h,\lambda}(s) - g_1(s) = \alpha_{t,h,\lambda}(s)(1 + o(h^{\frac{1}{2}})) + \beta_{t,h,\lambda}(s)(1 + o_p(1)).$$

Proof We only prove result (ii) here, because result (i) can be proved in a similar way. For simplicity, we suppress the symbol "(t,h,λ)" in $m_i^{(t,h,\lambda)}(s)$, which should not cause any confusion. By some algebraic manipulations, it can be checked that

$$\begin{aligned}
\widehat{g}_{t,h,\lambda}(s) - g_1(s) \;&=\; a_{0,t,\lambda} m_0^{-1}(s)[\phi_0(s) + \phi_2(s)] + a_{0,t,\lambda} m_0^{-1}(s) m_1(s)[m_2(s) - m_1^2(s) m_0^{-1}(s)]^{-1} \\
&\quad \cdot \{m_0^{-1}(s) m_1(s)[\phi_0(s) + \phi_2(s)] - \phi_1(s) - \phi_3(s)\} - g_1(s) \\
&= a_{0,t,\lambda} m_0^{-1}(s)\phi_0(s) + a_{0,t,\lambda} m_0^{-1}(s)[\phi_2(s) - a_{0,t,\lambda}^{-1} m_0(s) g_1(s) - a_{0,t,\lambda}^{-1} m_1(s) g_1'(s)] \\
&\quad + m_0^{-1}(s) m_1(s)\{g_1'(s) + a_{0,t,\lambda}[m_2(s) - m_1^2(s) m_0^{-1}(s)]^{-1} \cdot \\
&\quad [m_0^{-1}(s) m_1(s)(\phi_0(s) + \phi_2(s)) - \phi_1(s) - \phi_3(s)]\} \\
&=: \Delta_1 + \Delta_2 + \Delta_3.
\end{aligned}$$

By Taylor's expansions and by Condition (C5), it is straightforward that

$$\begin{aligned}
\Delta_1 &= \alpha_{0,t,h,\lambda}(s)\left(1 + O_p((c_{0,t,\lambda} h)^{-1/2}) + O(h)\right), \\
\Delta_2 &= \beta_{t,h,\lambda}(s)\left(1 + O_p((c_{0,t,\lambda} h)^{-1/2}) + O(h)\right) + O\left(\frac{a_{\tau,t,\lambda}}{a_{0,t,\lambda}} - 1\right).
\end{aligned}$$

By the facts that

$$\begin{aligned}
a_{0,t,\lambda}^{-1} m_1(s) &= \frac{\Gamma_2(s)}{\nu^2(s)} \int (u-s) K_h(u-s) du + O_p(c_{0,t,\lambda}^{-1/2} h^{1/2}) = O(h^2), \\
\phi_3(s) &= \frac{\Gamma_2(s)}{\nu^2(s)} [g_1(s) \int (u-s) K_h(u-s) du + h^2 g_1'(s)\eta_1] + O(h^3) + O_p(c_{0,t,\lambda}^{-1/2} h^{1/2}), \\
\phi_2(s) &= \frac{\Gamma_2(s)}{\nu^2(s)} g_1(s) + O(h), \qquad a_{0,t,\lambda}^{-1} m_2(s) = O(h^2),
\end{aligned}$$

we have

$$\Delta_3 = O_p(h^3) + O_p(c_{0,t,\lambda}^{-1/2} h^{1/2}).$$

By combining all the above results, Condition (C7), and the facts that $\alpha_{t,h,\lambda}(s) = O_p((c_{0,t,\lambda}h)^{-1/2})$ and $\beta_{t,h,\lambda}(s) = O_p(h^2)$, we can get result (ii) in the lemma. $\square$

Proof of Theorem 1

(i) Without loss of generality, we assume that $g_0 = 0$ (see related discussion in Section 2.3). By Lemma 1, we have

$$\begin{aligned}
T_{t,h,\lambda} &= \frac{c_{0,t,\lambda}}{n_0} \sum_{i=1}^{n_0} \frac{[\alpha_{t,h,\lambda}(s_i)]^2}{\nu^2(s_i)} (1 + o(h^{\frac{1}{2}})) \\
&= \frac{c_{0,t,\lambda}}{n_0} \sum_{i=1}^{n_0} \frac{\nu^2(s_i)}{a_{0,t,\lambda}^2 [\Gamma_2(s_i)]^2} \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j=1}^{n_k} \frac{[K_h(x_{kj} - s_i)]^2}{\nu^4(x_{kj})} [\varepsilon_{kj} + f_k(x_{kj})]^2 (1 + o(h^{\frac{1}{2}})) \\
&+ \frac{c_{0,t,\lambda}}{n_0} \sum_{i=1}^{n_0} \frac{\nu^2(s_i)}{a_{0,t,\lambda}^2 [\Gamma_2(s_i)]^2} \Bigg\{ \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j \neq l} \frac{[K_h(x_{kj} - s_i)][K_h(x_{kl} - s_i)]}{\nu^2(x_{kj})\nu^2(x_{kl})} [\varepsilon_{kj} + f_k(x_{kj})] \\
&\times [\varepsilon_{kl} + f_k(x_{kl})] + \sum_{k \neq k'} (1-\lambda)^{t-k} (1-\lambda)^{t-k'} \sum_{j,l} \frac{[K_h(x_{kj} - s_i)][K_h(x_{k'l} - s_i)]}{\nu^2(x_{kj})\nu^2(x_{k'l})} [\varepsilon_{kj} + f_k(x_{kj})] \\
&\times [\varepsilon_{k'l} + f_k(x_{k'l})] \Bigg\} (1 + o(h^{\frac{1}{2}})) =: (T_1 + T_2)(1 + o(h^{\frac{1}{2}}))
\end{aligned}$$

Note that, as $h \to 0$,

$$\begin{aligned}
T_1 &= \frac{c_{0,t,\lambda}}{a_{0,t,\lambda}^2} \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j=1}^{n_k} \frac{[\varepsilon_{kj} + f_k(x_{kj})]^2}{\nu^4(x_{kj})} \frac{1}{n_0} \sum_{i=1}^{n_0} \frac{\nu^2(s_i)}{[\Gamma_2(s_i)]^2} [K_h(x_{kj} - s_i)]^2 \\
&= \frac{c_{0,t,\lambda}\eta_2}{h a_{0,t,\lambda}^2} \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j=1}^{n_k} \frac{[\varepsilon_{kj} + f_k(x_{kj})]^2}{\nu^2(x_{kj})} \frac{\Gamma_1(x_{kj})}{[\Gamma_2(x_{kj})]^2} (1 + O(h) + O_p((n_0 h)^{-\frac{1}{2}})),
\end{aligned}$$

It is easy to see that

$$E(T_1) = \tilde{\mu}_h + o(h^{-\frac{1}{2}}), \quad \mathrm{Var}(T_1) = O\left[\frac{e_{0,t,\lambda}}{b_{0,t,\lambda}^2 h^2}\right] = o(h^{-1}).$$

Thus, we have

$$T_1 = E(T_1) + O_p(\sqrt{\mathrm{Var}(T_1)}) = \frac{\eta_2}{h} \int \frac{\Gamma_1(u)}{\Gamma_2(u)} du + o_p(h^{-1/2}).$$

Similarly, we have

$$
\begin{aligned}
T_2 = & \frac{c_{0,t,\lambda}}{a_{0,t,\lambda}^2 h} \Bigg\{ \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j \neq l} \frac{\Gamma_1(x_{kj})}{[\Gamma_2(x_{kj})]^2} K * K((x_{kj} - x_{kl})/h) \frac{\varepsilon_{kj}\varepsilon_{kl}}{\nu^2(x_{kj})} \\
& + \sum_{k \neq k'} (1-\lambda)^{t-k}(1-\lambda)^{t-k'} \sum_{j,l} \frac{\Gamma_1(x_{kj})}{[\Gamma_2(x_{kj})]^2} K * K((x_{kj} - x_{k'l})/h) \frac{[\varepsilon_{kj} + f_k(x_{kj})][\varepsilon_{k'l} + f_k(x_{k'l})]}{\nu^2(x_{kj})} \\
& + \sum_{k=1}^{t} (1-\lambda)^{2(t-k)} \sum_{j \neq l} \frac{\Gamma_1(x_{kj})}{[\Gamma_2(x_{kj})]^2} K * K((x_{kj} - x_{kl})/h) \frac{[\varepsilon_{kj} f_k(x_{kl}) + \varepsilon_{kl} f_k(x_{kj}) + f_k(x_{kj}) f_k(x_{kl})]}{\nu^2(x_{kj})} \Bigg\} \\
& \times (1 + O(h) + O_p((n_0 h)^{-\frac{1}{2}})) \\
& =: (T_{21} + T_{22} + T_{23})(1 + O(h) + O_p((n_0 h)^{-\frac{1}{2}})).
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
& E(T_{21}) = 0, \quad \mathrm{Var}(T_{21}) = O(e_{0,t,\lambda}/b_{0,t,\lambda}^2) = o(h^{-1}), \\
& E(T_{23}) = O(nh) = o(h^{-\frac{1}{2}}), \quad \mathrm{Var}(T_{23}) = O(e_{0,t,\lambda}/b_{0,t,\lambda}^2 n^2) = o(h^{-1}).
\end{aligned}
$$

Since $\sum_{j,l} \frac{\Gamma_1(x_{kj})}{[\Gamma_2(x_{kj})]^2} K * K((x_{kj} - x_{k'l})/h) \frac{[\varepsilon_{kj} + f_k(x_{kj})][\varepsilon_{k'l} + f_k(x_{k'l})]}{\nu^2(x_{kj})} =: \alpha_k$ are independent of each other for $k \neq k'$, $h^{1/2} T_{22}$ can be written as a symmetric quadratic function of α_k for $k = 1, \ldots, t$, with symmetric matrix $[\frac{c_{0,t,\lambda}}{a_{0,t,\lambda}^2 h^{\frac{1}{2}}}(1-\lambda)^{t-k}(1-\lambda)^{t-k'}]_{t \times t}$ which has vanishing diagonal elements. So, we can use Proposition 3.2 in de Jong (1987) to show the asymptotic normality of $h^{1/2} T_{22}$. Obviously, the expectation of T_{22} is zero. It can be checked that

$$
\mathrm{Var}(h^{1/2} T_{22}) = h \left(1 - \frac{e_{0,t,\lambda}}{b_{0,t,\lambda}^2} \right) \tilde{\sigma}_h^2 (1 + o(1)) = h \tilde{\sigma}_h^2 (1 + o(1)).
$$

Finally, by certain straightforward algebraic manipulations, we can verify that the moments of the elements in that quadratic form satisfy all the conditions given in Proposition 3.2 of de Jong (1987). Using this theorem and all the results above about T_1 and T_2, we have the result in Theorem 1-(i).

(ii) Using Lemma 1-(i), we have

$$
T_{t,h,\lambda} = \frac{c_{0,t,\lambda}}{n_0} \boldsymbol{\theta}^T \boldsymbol{\theta} (1 + o(h^{\frac{1}{2}})),
$$

where $\boldsymbol{\theta} = [\theta(s_1), \ldots, \theta(s_{n0})]^T = [\alpha_{t,h,\lambda}(s_1)/\nu(s_1), \ldots, \alpha_{t,h,\lambda}(s_{n_0})/\nu(s_{n0})]^T$. Next, we show the asymptotic multivariate normality of $\boldsymbol{\theta}$. To this end, it is sufficient to prove that, as

$c_{0,t,\lambda} \to \infty$, for any n_0-dimensional vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{n_0})$,

$$\sqrt{\frac{c_{0,t,\lambda}}{d_{0,t,\lambda}}}\boldsymbol{\omega}^T\boldsymbol{\theta} \sim AN(0, \boldsymbol{\omega}^T\boldsymbol{\Omega}\boldsymbol{\omega}).$$

Note that

$$\begin{aligned}\boldsymbol{\omega}^T\boldsymbol{\theta} &= \frac{1}{a_{0,t,\lambda}}\sum_{k=1}^{t}(1-\lambda)^{t-k}\sum_{i=1}^{n_0}\omega_i\frac{\nu(s_i)}{\Gamma_2(s_i)}\sum_{j=1}^{n_k}\frac{K_h(x_{kj}-s_i)}{\nu^2(x_{kj})}[\varepsilon_{kj}+f_k(x_{kj})]\\ &=: \frac{1}{a_{0,t,\lambda}}\sum_{k=1}^{t}M_k,\end{aligned}$$

where $M_k = \sum_{i=1}^{n_0}\omega_i R_k(s_i)$. Obviously, $E(\boldsymbol{\omega}^T\boldsymbol{\theta}|\boldsymbol{X}) = 0$. By some straightforward calculations, we have

$$\begin{aligned}&\mathrm{Cov}(R_k(s_i), R_k(s_j)|\boldsymbol{X})\\ &= (1-\lambda)^{2(t-k)}n_k^2\left[\frac{\nu(s_j)}{\nu(s_i)\Gamma_2(s_j)}\frac{K*K((s_i-s_j)/h)}{n_k h}+\frac{\tau(s_i,s_j)}{\nu(s_i)\nu(s_j)}\right](1+o_p(1))\\ &= (1-\lambda)^{2(t-k)}n_k^2\frac{\tau(s_i,s_j)}{\nu(s_i)\nu(s_j)}(1+o_p(1)),\end{aligned}$$

where the last equation comes from the condition that $nh \to \infty$. It follows that

$$\mathrm{Var}\left(\sqrt{\frac{c_{0,t,\lambda}}{d_{0,t,\lambda}}}\boldsymbol{\omega}^T\boldsymbol{\theta}\right) = \boldsymbol{\omega}^T\boldsymbol{\Omega}\boldsymbol{\omega}(1+o(1)).$$

Note that M_ks are independent of each other and that the moments of $\frac{t}{a_{0,t,\lambda}}M_k$ satisfy the Lindeberg conditions when Condition (C10) holds. Therefore, Theorem 1-(ii) follows immediately from the Lindeberg-Feller central limit theorem. □

Proof of Theorem 2

(i). Without loss of generality, we assume that $g_0 = 0$. Thus, $g_1 = \delta$. By Lemma 1-(ii), we have

$$\begin{aligned}T_{t,h,\lambda} = &\frac{c_{0,t,\lambda}}{n_0}\sum_{i=1}^{n_0}\frac{\alpha_{t,h,\lambda}^2(s_i)}{\nu^2(s_i)}(1+o(h^{\frac{1}{2}}))+\frac{c_{0,t,\lambda}}{n_0}\sum_{i=1}^{n_0}\frac{[\delta(s_i)+\beta_{t,h,\lambda}(s_i)]^2}{\nu^2(s_i)}(1+o_p(1))\\ &+\frac{2c_{0,t,\lambda}}{n_0}\sum_{i=1}^{n_0}\frac{\alpha_{t,h,\lambda}(s_i)\beta_{t,h,\lambda}(s_i)}{\nu^2(s_i)}(1+o_p(1))+\frac{2c_{0,t,\lambda}}{n_0}\sum_{i=1}^{n_0}\frac{\alpha_{t,h,\lambda}(s_i)\delta(s_i)}{\nu^2(s_i)}(1+o_p(1))\\ &=: T_1+T_2+(T_3+T_4)(1+o_p(1)).\end{aligned}$$

Obviously, T_1 is equivalent to $T_{t,h,\lambda}$ under the IC condition. It is straightforward to see that

$$\beta_{t,h,\lambda}(s) = \frac{h^2}{2}\delta''(s)\eta_1(1 + o_p(1)). \tag{A.5}$$

By this result, we have $T_2 = c_{0,t,\lambda}\zeta_\delta(1 + o_p(1))$, and

$$T_3 = \frac{h^2\eta_1 a_{0,t,\lambda}}{b_{0,t,\lambda}} \sum_{k=1}^{t}(1-\lambda)^{t-k} \sum_{j=1}^{n_k} \frac{\Gamma_1(x_{kj})}{\Gamma_2(x_{kj})\nu^2(x_{kj})}[\varepsilon_{kj} + f_k(x_{kj})]\delta''(x_{kj})(1 + o_p(1)).$$

Note that $\frac{1}{\sqrt{b_{0,t,\lambda}}}\sum_{k=1}^{t}(1-\lambda)^{t-k}\sum_{j=1}^{n_k}\frac{\Gamma_1(x_{kj})\delta''(x_{kj})}{\Gamma_2(x_{kj})\nu^2(x_{kj})}\varepsilon_{kj}$ is stochastically bounded, and

$$\frac{h^2\eta_1 a_{0,t,\lambda}}{b_{0,t,\lambda}} \sum_{k=1}^{t}(1-\lambda)^{t-k} \sum_{j=1}^{n_k} \frac{\Gamma_1(x_{kj})\delta''(x_{kj})}{\Gamma_2(x_{kj})\nu^2(x_{kj})} f_k(x_{kj}) = O_p(c_{0,t,\lambda}nh^5).$$

Thus, by condition (C8)-IV, we can conclude that $T_3 = o_p(h^{-1/2})$. Similarly,

$$\begin{aligned} T_4 &= \frac{2a_{0,t,\lambda}}{b_{0,t,\lambda}} \sum_{k=1}^{t}(1-\lambda)^{t-k} \sum_{j=1}^{n_k} \frac{\Gamma_1(x_{kj})\delta(x_{kj})}{\Gamma_2(x_{kj})\nu^2(x_{kj})}[\varepsilon_{kj} + f_k(x_{kj})] \\ &= O_p((c_{0,t,\lambda}n\int \delta^2(u)\frac{\Gamma_1(u)\gamma(u,u)}{\nu^2(u)}du)^{\frac{1}{2}}) = o_p(h^{-\frac{1}{2}}). \end{aligned}$$

By all these results and by Theorem 1, result (i) is proved.

(ii). This result follows directly from result (i).

(iii). Proof of this part is similar to that of Theorem 1-(ii), using Lemma 1-(ii) and result (A.5). So, it is omitted here. $\square$