analyze carcinogenesis experiments by Pike (1966), Peto et al. (1972), Peto and Lee (1973), and Williams (1978), to characterize early radiation response probabilities by Scott and Hahn (1980), and to model human disease-specific mortality by Juckett and Rosenberg (1990).

The Weibull distribution is characterized by two parameters, γ and λ. The value of γ determines the shape of the distribution curve and the value of λ determines its scaling. Consequently, γ and λ are called the shape and scale parameters, respectively. The relationship between the value of γ and survival time can be seen from Figure 6.4, which shows the hazard rate of the Weibull distribution with $\gamma = 0.5$, 1, 2, 4. When $\gamma = 1$, the hazard rate remains constant as time increases; this is the exponential case. The hazard rate increases when $\gamma > 1$ and decreases when $\gamma < 1$ as t increases. Thus, the Weibull distribution may be used to model the survival distribution of a population with increasing, decreasing, or constant risk. Examples of increasing and decreasing hazard rates are, respectively, patients with lung cancer and patients who undergo successful major surgery.

The probability density function and cumulative distribution functions are, respectively,

$$f(t) = \lambda\gamma(\lambda t)^{\gamma-1}e^{-(\lambda t)^{\gamma}} \qquad t \geq 0,\ \gamma,\ \lambda > 0 \tag{6.9}$$

and

$$F(t) = 1 - e^{-(\lambda t)^{\gamma}} \tag{6.10}$$

The survivorship function is therefore

$$S(t) = e^{-(\lambda t)^{\gamma}} \tag{6.11}$$

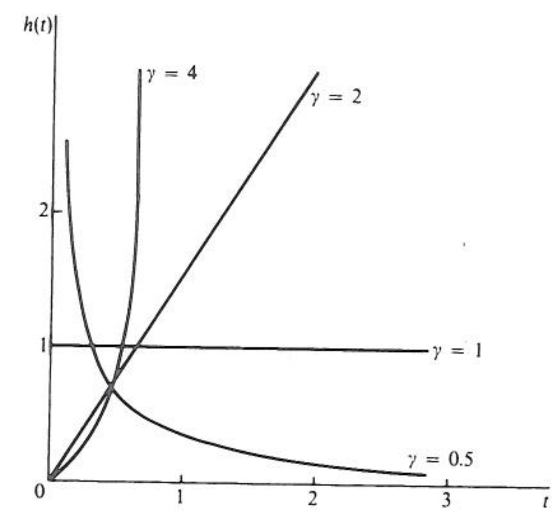


Figure 6.4 Hazard functions of Weibull distribution with $\lambda = 1$.

Peto et al. (1972), Peto
:rize early radiation re-
and to model human
; (1990).
o parameters, γ and λ.
tion curve and the value
are called the shape and
ween the value of γ and
hows the hazard rate of
n $\gamma = 1$, the hazard rate
nential case. The hazard
as t increases. Thus, the
irvival distribution of a
t risk. Examples of in-
vely, patients with lung
surgery.
e distribution functions

. $\lambda > 0$ (6.9)

(6.10)

(6.11)

1

5

ion with $\lambda = 1$.

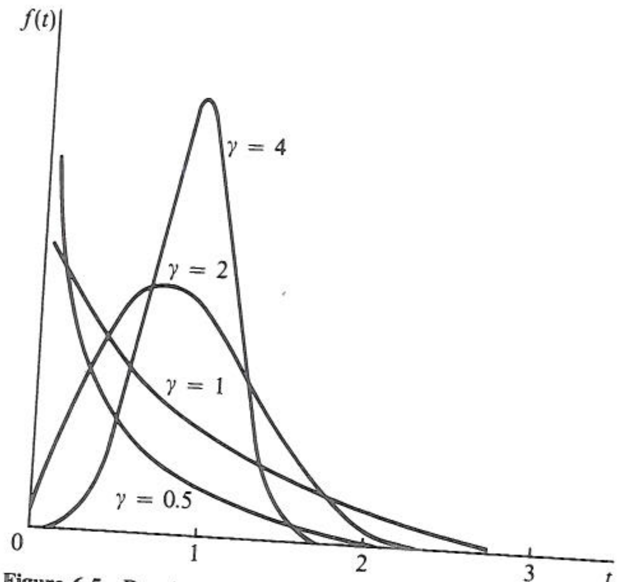


Figure 6.5 Density curves of Weibull distribution with $\lambda = 1$.

and the hazard function, the ratio of (6.9) to (6.11), is

$$h(t) = \lambda\gamma(\lambda t)^{\gamma-1} \tag{6.12}$$

Figure 6.5 gives the Weibull density function with scale parameter $\lambda = 1$ and several different values of the shape parameter γ.

For the survival curve, it is simple to plot the logarithm of $S(t)$,

$$\log_e S(t) = -(\lambda t)^\gamma \tag{6.13}$$

Figure 6.6 gives $\log_e S(t)$ for $\lambda = 1$ and $\gamma = 1, >1, <1$. When $\gamma = 1$, $\log_e S(t)$ is a straight line with negative slope. When $\gamma < 1$, negative aging, $\log_e S(t)$ decreases very slowly from zero and then approaches a constant value. When $\gamma > 1$, positive aging, $\log_e S(t)$ decreases sharply from zero as t increases.

The mean of the Weibull distribution is

$$\mu = \frac{\Gamma(1 + 1/\gamma)}{\lambda} \tag{6.14}$$

and the variance is

$$\sigma^2 = \frac{1}{\lambda^2}\left[\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right)\right] \tag{6.15}$$

where $\Gamma(\gamma)$ is the well-known gamma function defined as

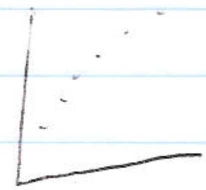
$$Y = \begin{cases} 0 & \text{cured} \\ 1 & \text{not cured.} \end{cases}$$

T: survival time

model:

$$P(T > t \mid Y = 1) = e^{-(\lambda t)^{\gamma}}$$

$$P(T > t \mid Y = 0, \underset{\text{age at entry}}{a}) = \frac{S_0(a+t)}{S_0(a)}.$$



$$P(Y = 0 \mid \underset{\text{covariate}}{x}) = \frac{1}{1 + e^{\alpha + \beta x}}$$

$$P(Y = 1 \mid x) = \frac{e^{\alpha + \beta x}}{1 + e^{\alpha + \beta x}}$$

Data: (t, δ, x, a, y)

$\delta = 0$ censored

$\delta = 1$ uncensored

t, δ, x, a are as before

$y = \begin{cases} 0 & \text{cured} \\ 1 & \text{not cured} \end{cases}$

The observation y could be missing.

The observation y is not equal to 0.

If $y = 1$ (died due to the disease)

the likelihood is:

$$L = \underbrace{\frac{e^{\alpha + \beta x_i}}{1 + e^{\alpha + \beta x_i}}}_{P(Y=1 \mid x)} \cdot \underbrace{e^{-(\lambda t)^{\gamma}} \cdot \left[\lambda \gamma (\lambda t)^{\gamma - 1}\right]^{\delta}}_{\text{Weibull}}$$

$\delta = 1$

$\delta = 0.$

If $y = 0$ the likelihood is

$$L = \left[\frac{1}{1+e^{\alpha+\beta x}} \frac{S_0(a+t)}{S_0(a)} + \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}} e^{-(\lambda t)^{\gamma}} \right]^{1-\delta}$$

$$\left[\frac{1}{1+e^{\alpha+\beta x}} f_0(a+t) + \frac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}} \lambda\gamma(\lambda t)^{\gamma-1} e^{-(\lambda t)^{\gamma}} \right]^{\delta}$$

$y=1.$

$P(y=1, T=t)$ $\qquad\qquad \delta=1.$

$= \underbrace{P(y=1)} \cdot \underbrace{P(T=t \mid y=1)}$

$\delta=0.$

$P(y=1, T>t)$

$= P(y=1)\ \underbrace{P(T>t \mid y=1)}.$