Convergence in Distribution and Uniform Convergence

Consider the following problem in statistics. Let $X_1, X_2, \dots,$ be an i.i.d. sequence of random variables with unknown mean, μ, and known variance σ^2. let $\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k$. Suppose we wish to approximate $P(|\bar{X}_n - \mu| \leq 1)$, which are useful in deriving asymptotic confidence intervals and hypothesis tests for μ. Then by the classical central limit theorem, we know that

$$Z_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} Z \stackrel{d}{=} N(0,1).$$

This means that $F_{Z_n}(t) \to F_Z(t)$ (pointwise) for all $t \in \mathbb{R}$. What we would like to say is that for large n

$$P(|\bar{X}_n - \mu| \leq 1) = P\left(\left|\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}\right| \leq \frac{1}{\sigma/\sqrt{n}}\right) = F_{Z_n}(\sqrt{n}/\sigma) \approx F_Z(\sqrt{n}/\sigma) = P(|Z| \leq \sqrt{n}/\sigma).$$

But it is not true in general that $F_{Z_n}(t) \to F_Z(t)$ (pointwise) for all $t \in \mathbb{R}$ implies that $F_{Z_n}(a_n) \approx F_Z(a_n)$ for large n. Even if $a_n \to a$, we cannot conclude that $F_{Z_n}(a_n) \to F_Z(a)$, without further assumptions as the following example shows.

Example 1 (Witch's Hat)**.** Consider the functions $f_n : [0,2] \mapsto \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} n^2 x, & 0 \leq x \leq 1/n \\ -n^2(x - 1/n) + n, & 1/n \leq x \leq 2/n \\ 0, & 2/n \leq x \leq 2 \end{cases}.$$

Then $f_n(x) \to 0$ (pointwise) for each each $x \in [0,2]$, but

$$f_n(1/n) = n \to \infty.$$

However, in this case, we have a much stronger mode of convergence occurring. It turns out that since F_Z is continuous, the convergence of F_{Z_n} to F_Z is actually uniform.

Theorem 1 (G. Polya)**.** *Let $\{F, F_n, n \geq 1\}$ be distribution functions. If F is continuous on $\mathbb{R}$ and $F_n(t) \to F(t)$ for all $t \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \sup_{-\infty \leq x \leq \infty} |F_n(x) - F(x)| = 0,$$

that is, $F_n \to F$ uniformly in x, $-\infty \leq x \leq \infty$.

Proof. Let $\varepsilon > 0$. Continuity of F ensures that points

$$-\infty = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} = \infty$$

may be chosen so that $F(x_{j+1}) - F(x_j) < \varepsilon$, $0 \leq j \leq k$. Since $F_n(t) \to F(t)$ pointwise for all $t \in \mathbb{R}$, we can choose an N such that for $n \geq N$

$$-\varepsilon < F_n(x_j) - F(x_j) < \varepsilon, j = 0, 1, \dots, k+1.$$

For $n \geq N$ and $x < \infty$, write $x_j \geq x < x_{j+1}$ for some $0 \leq j \leq k$, and then

$$F_n(x) - F(x) \leq F_n(x_{j+1}) - F(x_j)$$

$$\begin{aligned}&= F_n(x_{j+1}) - F(x_{j+1}) + F(x_{j+1}) - F(x_j)\\&< \varepsilon + \varepsilon = 2\varepsilon\end{aligned}$$

and

$$\begin{aligned}F_n(x) - F(x) &\geq F_n(x_j) - F(x_{j+1})\\&= F_n(x_j) - F(x_j) + F(x_j) - F(x_{j+1})\\&> -\varepsilon + -\varepsilon = -2\varepsilon.\end{aligned}$$

Thus, for $n \geq N$ and all $x \in \mathbb{R}$, $|F_n(x) - F(x)| < 2\varepsilon$. $\square$

Corollary 1. *Let $\{F, F_n, n \geq 1\}$ be distribution functions. If F is continuous on $\mathbb{R}$ and $F_n(t) \to F(t)$ for all $t \in \mathbb{R}$ and $\{a_n\}_n$ is any real sequences (bounded or not, having a limit or not), then*

$$\lim_{n\to\infty} |F_n(a_n) - F(a_n)| = 0.$$

Proof. $|F_n(a_n) - F(a_n)| \leq \sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \to 0.$ $\square$