Topics in Basic Analysis: Homework 1

1. For each of the following sets, find its supremum and infimum.

   a) $[2, 7]$

   b) $\cup_{n=1}^{\infty}[2n, 2n+1]$

   c) $\{1 - 1/3^n : n \in \mathbb{N}\}$

   d) $\cap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$

2. Let $S \subset \mathbb{R}$ be a nonempty, bounded set.

   a) Prove that $\inf S \leq \sup S$. (Hint: This should almost be obvious, your proof should be short.)

   b) What can you say about S if $\inf S = \sup S$?

3. Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets.

   a) Prove that if $S \subseteq T$, then
   $$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

   b) Prove that
   $$\sup(S \cup T) = \max\{\sup S, \sup T\}.$$
   Note that we are NOT assuming $S \subset T$ in part b).

4. Let $a, b \in \mathbb{R}$ be such that $a < b$. Using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, prove that there are infinitely many rational numbers between a an b.

5. Let $A, B \subset \mathbb{R}$ be nonempty, bounded sets, and let
   $$S = A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

   a) Prove that $\sup S = \sup A + \sup B$.

   b) Prove that $\inf S = \inf A + \inf B$.