Topics in Basic Analysis: Homework 1 Solutions

1. For each of the following sets, find its supremum and infimum.

   a) $[2, 7]$

   Solution. $\inf[2, 7] = 2$ and $\sup[2, 7] = 7$ □

   b) $\cup_{n=1}^{\infty}[2n, 2n + 1]$

   Solution. Note that

   $$\cup_{n=1}^{\infty}[2n, 2n + 1] = [2, 3] \cup [4, 5] \cup [6, 7] \cup \cdots ,$$

   so the infimum is 2 and the set is not bounded above, so the supremum is ∞. □

   c) $\{1 - 1/3^n : n \in \mathbb{N}\}$

   Solution. Since $\{1 - \frac{1}{3^n}, n \geq 1\}$ is an increasing sequence and converges to 1, the infimum is $1 - 1/3 = 2/3$ and the supremum is 1. □

   d) $\cap_{n=1}^{\infty}\left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$

   Solution. Note that

   $$\cap_{n=1}^{\infty}\left[-\frac{1}{n}, 1 + \frac{1}{n}\right] = [0, 1],$$

   so the infimum is 0 and the supremum is 1. □

2. Let $S \subset \mathbb{R}$ be a nonempty, bounded set.

   a) Prove that $\inf S \leq \sup S$. (Hint: This should almost be obvious, your proof should be short.)

   Solution. Let $\alpha = \inf S$ and $\beta = \sup S$. Then for all $x \in S$,

   $$\alpha \leq x \leq \beta.$$

   □

   b) What can you say about S if $\inf S = \sup S$?

   Solution. By part *a*), $\alpha \leq x \leq \beta$ for all $x \in S$, so if $\alpha = \beta$, then $x = \alpha = \beta$ for all $x \in S$. That is, S contains a single point, $S = \{\alpha\}$. □

3. Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets.

   a) Prove that if $S \subseteq T$, then

   $$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

   Solution. We have already proved the middle inequality in Q2a). Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets. Suppose that $S \subseteq T$. Then $x \in S \implies x \in T$, so $x \leq \sup T$ for all $x \in S$. This implies that $\sup S \leq \sup T$, since $\sup T$ is an upper bound for S. Similarly, $x \geq \inf T$ for all $x \in S$, which implies that $\inf S \geq \inf T$, since $\inf T$ is a lower bound for S. □

b) Prove that
$$\sup(S \cup T) = \max\{\sup S, \sup T\}.$$
Note that we are NOT assuming $S \subset T$ in part b).

Solution. We will prove this by showing that
$$\sup(S \cup T) \leq \max\{\sup S, \sup T\} \text{ and } \sup(S \cup T) \geq \max\{\sup S, \sup T\}.$$
Let $S, T \subset \mathbb{R}$ be nonempty, bounded sets. Note that
$$s \leq \sup S,\ \forall s \in S \text{ and } t \leq \sup T,\ \forall t \in T.$$
This implies that
$$x \leq \max\{\sup S, \sup T\}, \forall x \in S \cup T.$$
Thus, $\sup S \cup T \leq \max\{\sup S, \sup T\}$. Now, note that
$$x \leq \sup S \cup T,\ \forall x \in S \cup T.$$
Thus,
$$s \leq \sup S \cup T,\ \forall s \in S \implies \sup S \leq \sup S \cup T$$
and
$$t \leq \sup S \cup T,\ \forall t \in T \implies \sup T \leq \sup S \cup T.$$
Therefore,
$$\max\{\sup S, \sup T\} \leq \sup S \cup T.$$
□

4. Let $a, b \in \mathbb{R}$ be such that $a < b$. Using the denseness of $\mathbb{Q}$ in $\mathbb{R}$, prove that there are infinitely many rational numbers between a an b.

Solution. Let $a, b \in \mathbb{R}$ be such that $a < b$. Suppose that there are only finitely many rational numbers between a and b. Since there are finitely many, we can list them $\{r_1, r_2, \ldots, r_n\}$. Let $q \in \{r_1, r_2, \ldots, r_n\}$ be the smallest such that $q > a$. Then by the density of $\mathbb{Q}$, $\exists r_{n+1} \in \mathbb{Q}$ such that $a < r_{n+1} < q$, but this contradicts our assumption that there are only n such rational points between a and b. □

5. Let $A, B \subset \mathbb{R}$ be nonempty, bounded sets, and let
$$S = A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

a) Prove that $\sup S = \sup A + \sup B$.

Solution. Again, we will prove this by showing that
$$\sup S \leq \sup A + \sup B \text{ and } \sup S \geq \sup A + \sup B.$$
Since
$$a \leq \sup A,\ \forall a \in A \text{ and } b \leq \sup B,\ \forall b \in B,$$
we have
$$a + b \leq \sup A + \sup B,\ \forall a \in A, b \in B.$$

This implies that $\sup A + \sup B$ is an upper bound for S, so

$$\sup S \leq \sup A + \sup B.$$

Now, note that

$$\begin{aligned} a + b \leq \sup S,\ \forall a \in A, b \in B &\implies a \leq \sup S - b,\ \forall a \in A, b \in B \\ &\implies \sup A \leq \sup S - b,\ \forall b \in B \\ &\implies b \leq \sup S - \sup A,\ \forall b \in B \\ &\implies \sup B \leq \sup S - \sup A \\ &\implies \sup A + \sup B \leq \sup S. \end{aligned}$$

□

b) Prove that $\inf S = \inf A + \inf B$.

Solution. The proof is nearly identical to part a). □