Topics in Basic Analysis: Homework 2

1. Prove that $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$.
2. Determine the limits of the following sequences and prove your claim.
   a) $a_n = \dfrac{4n+3}{7n-5}$, $n \geq 1$.
   b) $s_n = \frac{1}{n}\sin n$, $n \geq 1$.
3. Prove the following claim: If $(a_n)_n, (b_n)_n$ and $(s_n)_n$ are reals sequences such that $a_n \leq s_n \leq b_n$ for all $n \geq 1$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, then $\lim_{n\to\infty} s_n = s$.
4. Prove that $\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$. Hint: Consider multiplying by $1 = \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$.
5. Let $(s_n)_n$ and $(t_n)_n$ be real sequences, and suppose $\exists N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for $n \geq N_0$. Prove the following statements.
   a) If $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} t_n = \infty$.
   b) If $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$.
   c) If $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exists, then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.
6. Let $(s_n)_n$ and $(t_n)_n$ be real sequences. Prove the following statements:
   a) If $\lim_{n\to\infty} s_n = \infty$ and $\inf_{n\in\mathbb{N}} t_n > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.
   b) If $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.
   c) If $\lim_{n\to\infty} s_n = \infty$ and $(t_n)_n$ is bounded, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.
7. Let $(s_n)_n$ be a real sequence and assume that $s_n \neq 0$ for all $n \geq 1$. Suppose that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ exists.
   a) Prove that if $L < 1$, then $\lim_{n\to\infty} s_n = 0$. Hint: Select a so that $L < a < 1$, and obtain an N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show that $|s_n| < a^{n-N}|s_N|$ for $n > N$.
   b) Show that if $L > 1$, then $\lim_{n\to\infty} |s_n| = \infty$. Hint: Apply part a) to the sequence $t_n = 1/|s_n|$.