Topics in Basic Analysis: Homework 2 Solutions

1. Prove that $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$.

   Solution. Note that for $n \geq 1$,

$$\begin{aligned}\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| &= \left|\frac{6n-3-6n-4}{9n+6}\right| \\ &= \left|\frac{-7}{9n+6}\right| \\ &\leq \frac{9}{9n} \\ &= \frac{1}{n} \to 0.\end{aligned}$$

   Thus $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$ by the squeeze theorem. □

2. Determine the limits of the following sequences and prove your claim.

   a) $a_n = \dfrac{4n+3}{7n-5}$, $n \geq 1$.

   Solution. From calculus, we know that $a_n \to \frac{4}{7}$. To see this, note that $7n-5 \geq 2n \iff n \geq 1$, so for $n \geq 1$,

$$\begin{aligned}\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| &= \left|\frac{28n+21-28n+20}{7(7n-5)}\right| \\ &= \left|\frac{41}{7(7n-5)}\right| \\ &\leq \frac{42}{7(2n)} \\ &= \frac{3}{n} \to 0.\end{aligned}$$

   Thus $a_n \to \frac{4}{7}$ by the squeeze theorem. □

   b) $s_n = \frac{1}{n}\sin n$, $n \geq 1$,

   Solution. Note that for $n \geq 1$.

$$|s_n| = \left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \to 0.$$

   Thus, $s_n \to 0$ by the squeeze theorem. □

3. Prove the following claim: If $(a_n)_n, (b_n)_n$ and $(s_n)_n$ are reals sequences such that $a_n \leq s_n \leq b_n$ for all $n \geq 1$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, then $\lim_{n\to\infty} s_n = s$.

   Solution. Let $\varepsilon > 0$. Since $b_n \to 0$, $\exists N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies |b_n - s| < \varepsilon \implies b_n < s + \varepsilon.$$

Similarly, since $a_n \to s$, $\exists N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies |a_n - s| < \varepsilon \implies s - \varepsilon < a_n.$$

Thus, for $n \geq \max\{N_2, N_2\}$

$$s - \varepsilon < a_n \leq s_n \leq b_n < s + \varepsilon \implies |s_n - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $s_n \to s$. □

4. Prove that $\lim_{n\to\infty} \sqrt{n^2+n} - n = \frac{1}{2}$. Hint: Consider multiplying by $1 = \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$.

Solution. First, note that $n \geq 1$,

$$\begin{aligned}\sqrt{n^2+n} - n &= \sqrt{n^2+n} - n \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\ &\to \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.\end{aligned}$$

Note, that we have used two facts in the last line:

- $s_n \to s \implies \sqrt{s_n} \to \sqrt{s}$ if $s_n, s \geq 0$.
- $s_n \to s \implies 1/s_n \to 1/s$ if $s_n, s \neq 0$.

The second point was proven in class. The first was mentioned and used but not proven. To see why this is true, we can argue as follows. Let $\varepsilon > 0$. Note that

$$\sqrt{s_n} - \sqrt{s} = \sqrt{s_n} - \sqrt{s} \cdot \frac{\sqrt{s_n}+\sqrt{s}}{\sqrt{s_n}+\sqrt{s}} = \frac{s_n - s}{\sqrt{s_n}+\sqrt{s}}.$$

Since $s_n \to s$, $\exists N_1 \in \mathbb{N}$ such that $s_n > s/4$ for $n \geq N_1$, and $\exists N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $|s_n - s| < \frac{3}{2}\sqrt{s}\varepsilon$. Then for $n \geq \max N_1, N_2$,

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n}+\sqrt{s}} \leq \frac{\frac{3}{2}\sqrt{s}\varepsilon}{\frac{3}{2}\sqrt{s}} = \varepsilon.$$

□

5. Let $(s_n)_n$ and $(t_n)_n$ be real sequences, and suppose $\exists N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for $n \geq N_0$. Prove the following statements.

a) If $\lim_{n\to\infty} s_n = \infty$, then $\lim_{n\to\infty} t_n = \infty$.

Solution. Let $M > 0$. Since $s_n \to \infty$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $M \leq s_n$. Then, for $n \geq N, N_0$

$$M \leq s_n \leq t_n.$$

Since M was arbitrary, $t_n \to \infty$. □

b) If $\lim_{n\to\infty} t_n = -\infty$, then $\lim_{n\to\infty} s_n = -\infty$.

Solution. Let $M < 0$. Since $t_n \to -\infty$, $\exists N \in \mathbb{N}$ such that $n \geq N$ implies $t_n \leq M$. Then, for $n \geq N, N_0$

$$s_n \leq t_n \leq M.$$

Since M was arbitrary, $s_n \to -\infty$. □

c) If $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exists, then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.

Solution. The inequality is already proven in parts a) and b) if either $s_n \to \infty$ or $t_n \to -\infty$. It is trivially true if $s_n \to -\infty$ or $t_n \to \infty$. The only case that remains is if both $s_n \to s \in \mathbb{R}$ and $t_n \to t \in \mathbb{R}$. Then

$$0 \leq t_n - s_n, \forall n \geq 1 \text{ and } t_n - s_n \to t - s$$

implies that $t - s \geq 0 \iff t \geq s$. □

6. Let $(s_n)_n$ and $(t_n)_n$ be real sequences. Prove the following statements:

a) If $\lim_{n\to\infty} s_n = \infty$ and $\inf_{n\in\mathbb{N}} t_n > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.

Solution. Let $M > 0$. Let $K = \inf_{n\in\mathbb{N}} t_n$. Then $t_n \geq K$ for all $n \geq 1$. Since $s_n \to \infty$, choose $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \geq M - K$. Then $n \geq N$ implies

$$s_n + t_n \geq M - K + K = M.$$

□

b) If $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.

Solution. We need to consider two cases: (1) $\lim_{n\to\infty} t_n = t \in \mathbb{R}$ and (2) $\lim_{n\to\infty} t_n = \infty$.

Case 1: Let $M > 0$. Choose $N_1 \in \mathbb{N}$ such that $n \geq N_2$ implies

$$|t_n - t| \leq \frac{|t|}{2}.$$

Then as we argued in class in Proposition 2.6 part d), $|t_n| > |t|/2$ for $n \geq N_1$. Next, choose $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies $s_n \geq M - |t|/2$. Then for $n \geq \max\{N_1, N_2\}$,

$$s_n + t_n \geq M - \frac{|t|}{2} + \frac{|t|}{2} = M.$$

Case 2: Let $M > 0$. Choose $N_1 \in \mathbb{N}$ such that

$$n \geq N_1 \implies t_n \geq \frac{M}{2},$$

and choose $N_2 \in \mathbb{N}$ such that

$$n \geq N_2 \implies s_n \geq \frac{M}{2}.$$

Then for $n \geq \max\{N_1, N_2\}$

$$s_n + t_n \geq \frac{M}{2} + \frac{M}{2} = M.$$

□

c) If $\lim_{n\to\infty} s_n = \infty$ and $(t_n)_n$ is bounded, then $\lim_{n\to\infty}(s_n + t_n) = \infty$.

Solution. Let $M > 0$, and let $K > 0$ be such that $|t_n| \leq K$ for all $n \geq 1$. Then $t_n \geq -K$ for all $n \geq 1$. Since $s_n \to \infty$, choose $N \in \mathbb{N}$ such that $n \geq N$ implies $s_n \geq M - K$. Then $n \geq N$ implies

$$s_n + t_n \geq M - K + K = M.$$

□

7. Let $(s_n)_n$ be a real sequence and assume that $s_n \neq 0$ for all $n \geq 1$. Suppose that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ exists.

a) Prove that if $L < 1$, then $\lim_{n\to\infty} s_n = 0$. Hint: Select a so that $L < a < 1$, and obtain an N so that $|s_{n+1}| < a|s_n|$ for $n \geq N$. Then show that $|s_n| < a^{n-N}|s_N|$ for $n > N$.

Solution. Suppose that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L < 1$. Let $L < a < 1$. Then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies

$$\left|\frac{s_{n+1}}{s_n}\right| < a.$$

Then by a similar induction argument used in the proof of Theorem 2.2, we have for $n > N$

$$|s_n| < a^{n-N}|s_N| = a^n \left(\frac{s_N}{a}\right)^N.$$

Since $0 \leq L < a < 1$, $a^n \to 0$, which implies that $s_n \to 0$ by the squeeze theorem. □

b) Show that if $L > 1$, then $\lim_{n\to\infty} |s_n| = \infty$. Hint: Apply part a) to the sequence $t_n = 1/|s_n|$.

Solution. Note that with $t_n = 1/|s_n|$,

$$\lim_{n\to\infty} \left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty} \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L} < 1,$$

since $L > 1$. Thus, by part a), $t_n \to 0$. Since $|s_n| > 0$ and $1/|s_n| \to 0$, $|s_n| \to \infty$ by Proposition 2.8 part b) of the course notes. □