Topics in Basic Analysis: Homework 3

1. Determine if the following sequences are increasing, decreasing, or neither, and if the sequence is bounded.

   a) $\frac{1}{n}$
   b) $\frac{(-1)^n}{n^2}$
   c) $\sin\left(\frac{n\pi}{7}\right)$
   d) $\frac{n}{3^n}$

2. Let $(s_n)_n$ be a sequence such that
$$|s_{n+1} - s_n| < 2^{-n}, \ \forall n \in \mathbb{N}.$$

   a) Prove that $(s_n)_n$ is a Cauchy sequence and hence converges.
   b) Is is still true that $(s_n)_n$ is Cauchy if we only assume that
$$|s_{n+1} - s_n| < \frac{1}{n}, \ \forall n \in \mathbb{N}?$$

3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

   a) Find s_2, s_3, and s_4.
   b) Use induction to show that $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.
   c) Show that $(s_n)_n$ is decreasing.
   d) Show that $\lim_{n\to\infty} s_n = s$ exists and find s.

4. For each of the following sequence:
$$s_n = \cos\left(\frac{n\pi}{3}\right) \quad t_n = \frac{3}{4n+1} \quad u_n = \left(\frac{1}{2}\right)^n \quad v_n = (-1)^n + \frac{1}{n}$$

   a) Give its set of subsequential limit points.
   b) Give its lim sup and lim inf.

5. Let $(s_n)_n$ and $(t_n)_n$ be sequences, and suppose that there exists and $N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n \geq N_0$. Show that $\liminf s_n \leq \liminf t_n$ and $\limsup s_n \leq \limsup t_n$. (Hint: Consider the definition of lim inf and lim sup and HW2 problem 5c).

6. Let $(s_n)_n$ and $(t_n)_n$ be bounded real sequences. Show that
$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

7. Let $(s_n)_n$ and $(t_n)_n$ be bounded real sequences. Show that
$$\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).$$

8. Let $(s_n)_n$ be a real sequence and define $\sigma_n = \frac{1}{n}\sum_{i=1}^{n} s_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$.

   a) Show that
$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$
   Hint: For the third inequality, show first that $M > N$ implies
$$\sup_{n\geq M} \sigma_n \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup_{n\geq N} s_n.$$
   b) Show that if $\lim_{n\to\infty} s_n$ exists, then $\lim_{n\to\infty} \sigma_n$ exists and $\lim s_n = \lim \sigma_n$.