Topics in Basic Analysis: Homework 3 Solutions

1. Determine if the following sequences are increasing, decreasing, or neither, and if the sequence is bounded.

   a) $\frac{1}{n}$

   Solution. Decreasing and bounded. Note that for $x, y > 0$

   $$\frac{1}{x} < \frac{1}{y} \iff y < x.$$

   Alternatively, note that the function $f(x) = 1/x$ has derivative $f'(x) = -1/x^2 < 0$ for all $x > 0$, so $1/n = f(n)$ is decreasing. Since $1/n \to 0$, the sequence is bounded. □

   b) $\frac{(-1)^n}{n^2}$

   Solution. Neither and bounded. Note that $\frac{(-1)^n}{n^2} \to 0$, so it is bounded. However, the sequence alternates between positive and negative values, so it neither increasing nor decreasing for all $n \geq 1$. □

   c) $\sin\left(\frac{n\pi}{7}\right)$

   Solution. Neither and bounded. Note that $\left|\sin\left(\frac{n\pi}{7}\right)\right| \leq 1$, so the sequence is bounded. Since the sin function oscillates, it is neither increasing nor decreasing for all $n \geq 1/$ □

   d) $\frac{n}{3^n}$

   Solution. Decreasing and bounded. Since $n/3^n \to 0$ it is bounded. To see that it is decreasing, consider $f(x) = x/3^x$. Then f is decreasing if and only if $\ln f(x) = \ln x - x \ln 3$ is decreasing. Since

   $$\frac{d}{dx}\left[\ln x - x \ln 3\right] = \frac{1}{x} - \ln 3 < 0 \iff x > \frac{1}{\ln 3} \approx 0.91.$$

   Thus $\left(\frac{n}{3^n}\right)_n$ is decreasing for $n \geq 1$. □

2. Let $(s_n)_n$ be a sequence such that

   $$|s_{n+1} - s_n| < 2^{-n}, \ \forall n \in \mathbb{N}.$$

   a) Prove that $(s_n)_n$ is a Cauchy sequence and hence converges.

   Solution. Let $\varepsilon > 0$. Note that $\sum_{n=1}^{\infty} 2^{-n} < \infty$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies

   $$\sum_{k=n}^{\infty} 2^{-k} < \varepsilon.$$

   Then for $m > n \geq N$

   $$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - \cdots - s_{n+1} + s_{n+1} - s_n|$$

$$\begin{aligned}
&\le \sum_{k=n}^{m-1} |s_{k+1} - s_k| \\
&< \sum_{k=n}^{m-1} 2^{-k} \\
&\le \sum_{k=n}^{\infty} 2^{-k} \\
&< \varepsilon.
\end{aligned}$$

□

b) Is is still true that $(s_n)_n$ is Cauchy if we only assume that

$$|s_{n+1} - s_n| < \frac{1}{n}, \ \forall n \in \mathbb{N}?$$

Solution. No. Consider the sequence $s_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \ge 1$. Then for all $n \ge 1$

$$|s_{n+1} - s_n| = \left| \sum_{k=1}^{n+1} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \right| = \frac{1}{n+1} < \frac{1}{n},$$

but $s_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to infinity and hence is not Cauchy. □

3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \ge 1$.

a) Find s_2, s_3, and s_4.

Solution.

$$s_2 = \frac{1}{3}(1+1) = \frac{2}{3} \quad s_3 = \frac{1}{3}\left(\frac{2}{3} + 1\right) = \frac{5}{9} \quad s_4 = \frac{1}{3}\left(\frac{5}{9} + 1\right) = \frac{14}{27}.$$

□

b) Use induction to show that $s_n > \frac{1}{2}$ for all $n \in \mathbb{N}$.

Solution. Note that $s_1 = 1 > \frac{1}{2}$. Now suppose that $s_k > \frac{1}{2}$ for some $k \ge 1$. Then

$$s_{k+1} = \frac{1}{3}(s_k + 1) > \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4} > \frac{1}{2}.$$

Then by induction $s_n > \frac{1}{2}$ for all $n \ge 1$. □

c) Show that $(s_n)_n$ is decreasing.

Solution. Note that $s_2 = \frac{5}{9} \ge \frac{2}{3} = s_1$. Now, suppose that $s_k \ge s_{k-1}$ for some $k \ge 2$. Then,

$$s_{k+1} = \frac{1}{3}(s_k + 1) \ge \frac{1}{3}(s_{k-1} + 1) = s_k.$$

Then by induction $s_{k+1} \ge s_k$ for all $k \ge 1$, i.e. $(s_n)_n$ is decreasing. □

d) Show that $\lim_{n\to\infty} s_n = s$ exists and find s.

Solution. Since $(s_n)_n$ is decreasing and bounded below $\lim_{n\to\infty} s_n = s$ exists and is finite. Since $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 2$, we have

$$\lim_{n\to\infty} s_{n+1} = \frac{1}{3}\left(\lim_{n\to\infty} s_n + 1\right) \implies s = \frac{1}{3}(s+1) \implies s = \frac{1}{2}.$$

□

4. For each of the following sequence:

$$s_n = \cos\left(\frac{n\pi}{3}\right) \quad t_n = \frac{3}{4n+1} \quad u_n = \left(\frac{1}{2}\right)^n \quad v_n = (-1)^n + \frac{1}{n}$$

a) Give its set of subsequential limit points.

Solution. Note that

$$(s_n)_n = \left\{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots\right\},$$

so the set of subsequential limit points of $(s_n)_n$ is $\left\{1, \frac{1}{2}, -\frac{1}{2}, -1\right\}$.

Since $t_n \to 0$, the only subsequential limit point is $\{0\}$.

Since $u_n \to 0$, the only subsequential limit point is $\{0\}$.

Since $\frac{1}{n} \to 0$ and $(-1)^n = \{-1, 1, -1, 1, \ldots\}$, the subsequential limit points of of v_n are $\{-1, 1\}$.

□

b) Give its lim sup and lim inf.

Solution.

$$\liminf_{n\to\infty} s_n = -1,\ \limsup_{n\to\infty} s_n = 1, \quad \text{and} \quad \liminf_{n\to\infty} v_n = -1,\ \limsup_{n\to\infty} v_n = 1.$$

$$\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = 0, \quad \text{and} \quad \liminf_{n\to\infty} u_n = \limsup_{n\to\infty} u_n = 0,$$

□

5. Let $(s_n)_n$ and $(t_n)_n$ be sequences, and suppose that there exists and $N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n \geq N_0$. Show that $\liminf s_n \leq \liminf t_n$ and $\limsup s_n \leq \limsup t_n$. (Hint: Consider the definition of lim inf and lim sup and HW2 problem 5c).

Solution. Let $N \geq N_0$, then $s_n \leq t_n$ for all $n \geq N$, and

$$\inf_{k\geq N}(t_n - s_n) \geq 0.$$

Note that for all $n \geq N$

$$t_n = t_n - s_n + s_n \geq \inf_{k\geq N}(t_k - s_k) + \inf_{k\geq N} s_k \geq \inf_{k\geq N} s_k \implies \inf_{k\geq N} t_k \geq \inf_{k\geq N} s_k.$$

Since $\inf_{k\geq N} t_k \geq \inf_{k\geq N} s_k$ for all $N \geq N_0$,

$$\liminf_{k\to\infty} s_k = \lim_{N\to\infty} \inf_{k\geq N} s_k \leq \lim_{N\to\infty} \inf_{k\geq N} t_k = \liminf_{k\to\infty} t_k.$$

Similarly, $s_n \leq t_n$ for all $n \geq N$ implies

$$\sup_{k\geq N}(s_k - t_k) \leq 0.$$

Then for all $n \geq N$

$$s_n = s_n - t_n + t_n \leq \sup_{k\geq N}(s_k - t_k) + \sup_{k\geq N} t_k \leq \sup_{k\geq N} t_k \implies \sup_{k\geq N} s_k \leq \sup_{k\geq N} t_k.$$

Therefore,

$$\limsup_{k\to\infty} s_k = \lim_{N\to\infty} \sup_{k\geq N} s_k \leq \lim_{N\to\infty} \sup_{k\geq N} t_k = \limsup_{k\to\infty} t_k.$$

□

6. Let $(s_n)_n$ and $(t_n)_n$ be bounded real sequences. Show that

$$\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n.$$

Solution. Since $(s_n)_n$ and $(t_n)_n$ are bounded, $(s_n + t_n)_n$ is bounded, so

$$\sup_{k\geq N} s_k, \ \sup_{k\geq N} t_k, \quad \text{and} \quad \sup_{k\geq N}(s_k + t_k)$$

exists and are finite for all $N \in \mathbb{N}$. Let $N \in \mathbb{N}$. Then for all $n \geq N$

$$s_n \leq \sup_{k\geq N} s_k \quad \text{and} \quad t_n \leq \sup_{k\geq N} t_k,$$

so

$$s_n + t_n \leq \sup_{k\geq N} s_k + \sup_{k\geq N} t_k, \ \forall n \geq N.$$

Therefore

$$\sup_{k\geq N}(s_k + t_k) \leq \sup_{k\geq N} s_k + \sup_{k\geq N} t_k,$$

for all $N \geq \mathbb{N}$, which implies

$$\limsup_{k\to\infty}(s_k + t_k) = \lim_{N\to\infty} \sup_{k\geq N}(s_k + t_k) \leq \lim_{N\to\infty} \sup_{k\geq N} s_k + \lim_{N\to\infty} \sup_{k\geq N} t_k = \limsup_{k\to\infty} s_k + \limsup_{k\to\infty} t_k.$$

□

7. Let $(s_n)_n$ and $(t_n)_n$ be bounded real sequences. Show that

$$\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n).$$

Solution. Let $N \in \mathbb{N}$. Note that for $n \geq N$

$$s_n \leq \sup_{k\geq N} s_k \text{ and } t_n \leq \sup_{k\geq N} t_k \implies s_n t_n \leq \sup_{k\geq N} s_k \sup_{k\geq N} t_k$$

for all $n \geq N$. Therefore,

$$\sup_{k\geq N}(s_k t_k) \leq \sup_{k\geq N} s_k \sup_{k\geq N} t_k,$$

which implies

$$\limsup_{k\to\infty}(s_k t_k) = \lim_{N\to\infty} \sup_{k\geq N}(s_k t_k) \leq \lim_{N\to\infty} [\sup_{k\geq N} s_k \sup_{k\geq N} t_k] = \lim_{N\to\infty} \sup_{k\geq N} s_k \lim_{N\to\infty} \sup_{k\geq N} t_k = \limsup_{k\to\infty} s_k \limsup_{k\to\infty} t_k.$$

□

8. Let $(s_n)_n$ be a real sequence and define $\sigma_n = \frac{1}{n}\sum_{i=1}^n s_n = \frac{1}{n}(s_1 + s_2 + \ldots + s_n)$.

   a) Show that

   $$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

   Hint: For the third inequality, show first that $M > N$ implies

   $$\sup_{n\geq M} \sigma_n \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup_{n\geq N} s_n.$$

   Solution. Let $N \in \mathbb{N}$. Then for $n \geq M > N$

   $$\begin{aligned} \sigma_n &= \frac{1}{n}(s_1 + s_2 + \cdots s_n) \\ &= \frac{1}{n}(s_1 + \cdots + s_N) + \frac{1}{n}(s_{N+1} + \cdots s_n) \\ &\leq \frac{1}{M}(s_1 + \cdots + s_N) + \frac{n-N}{n} \sup_{k\geq N} s_k \\ &\leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup_{k\geq N} s_k. \end{aligned}$$

   This implies that for all $M > N$

   $$\sup_{k\geq M} \sigma_k \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup_{k\geq N} s_k.$$

   Therefore,

   $$\limsup_{k\to\infty} \sigma_k = \lim_{M\to\infty} \sup_{k\geq M} \sigma_k \leq \lim_{M\to\infty} \left[\frac{1}{M}(s_1 + \cdots + s_N) + \sup_{k\geq N} s_k\right] = \sup_{k\geq N} s_k.$$

   Since $N \in \mathbb{N}$ was arbitrary,

   $$\limsup_{k\to\infty} \sigma_k \leq \lim_{N\to\infty} \sup_{k\geq N} s_k = \limsup_{k\to\infty} s_k.$$

   By the same argument using $-\sigma_n = \frac{1}{n}(-s_1 + -s_2 + \cdots + -s_n)$, we also have

   $$\limsup_{k\to\infty}(-\sigma_k) \leq \limsup_{k\to\infty}(-s_k).$$

   Using the fact that $\limsup_{k\to\infty}(-x_k) = -\liminf_{k\to\infty} x_k$, we obtain

   $$\liminf_{k\to\infty} s_k \leq \liminf_{k\to\infty} \sigma_k.$$

□

b) Show that if $\lim_{n\to\infty} s_n$ exists, then $\lim_{n\to\infty} \sigma_n$ exists and $\lim s_n = \lim \sigma_n$.

Solution. If $\lim_{n\to\infty} s_n = s \in \mathbb{R} \cup \{-\infty, \infty\}$ exists, then by part a)

$$s = \liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n = s,$$

which implies that $\lim_{n\to\infty} \sigma_n = s$. □