Topics in Basic Analysis: Homework 4

1. Prove that the following sets have an empty interior.

   a) $\{\frac{1}{n} | n \in \mathbb{N}\}$

   b) $\mathbb{Q}$

2. Find the closure of the following sets.

   a) $\mathbb{Q}$

   b) $\{r \in \mathbb{Q} : r^2 < 2\}$

3. Determine if the following series converge or diverge. Be sure to justify your answers.

   a) $\sum_{n=1}^{\infty} \dfrac{n^4}{2^n}$

   b) $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

   c) $\sum_{n=1}^{\infty} \dfrac{n!}{n^4+3}$

   d) $\sum_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2}$

   e) $\sum_{n=2}^{\infty} \dfrac{1}{[n+(-1)^n]^2}$

   f) $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$

   g) $\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n}\ln n}$

   h) $\sum_{n=4}^{\infty} \dfrac{1}{n(\ln n)(\ln\ln n)}$

   i) $\sum_{n=2}^{\infty} \dfrac{\ln n}{n^2}$

4. Prove the following: If $\sum_{n=1}^{\infty} |a_n|$ converges and $(b_n)_n$ is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

5. Prove the following: If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum_{n=1}^{\infty} a_n^p$ converges.

6. Prove the following: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, then $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges. Hint: Show that $\sqrt{a_n b_n} \le a_n + b_n$ for all $n \ge 1$.