Topics in Basic Analysis: Homework 4 Solutions

1. Prove that the following sets have an empty interior.

   a) $\left\{\frac{1}{n} | n \in \mathbb{N}\right\}$

   Solution. Suppose that the interior is not empty. Let $x \in \left\{\frac{1}{n} | n \in \mathbb{N}\right\}^\circ$. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) \subseteq \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$. Then $1/n_0 \in B(x, \varepsilon)$ for some $n_0 \in \mathbb{N}$. Since $B(x, \varepsilon)$ is open, $\exists \rho > 0$ such that $B(1/n_0, \rho) \subseteq B(x, \varepsilon)$, but $B(1/n_0, \rho) = (1/n_0 - \rho, 1/n_0 + \rho) \nsubseteq \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$. □

   b) $\mathbb{Q}$

   Solution. Suppose $\mathbb{Q}^\circ \neq \emptyset$. Let $x \in \mathbb{Q}^\circ$. Then $\exists \varepsilon > 0$ such that $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon) \subseteq \mathbb{Q}$. By the denseness of the irrationals, $\exists t \in \mathbb{R} \setminus \mathbb{Q}$ such that

   $$x - \varepsilon < t < x \implies t \in B(x, \varepsilon) \subseteq \mathbb{Q},$$

   but $t \notin \mathbb{Q}$. Hence, $\mathbb{Q}^\circ = \emptyset$. □

2. Find the closure of the following sets.

   a) $\mathbb{Q}$

   Solution. Recall that $\bar{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}'$. Since $\mathbb{Q}$ are dense in $\mathbb{R}$, every $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$, so $\bar{\mathbb{Q}} = \mathbb{R}$. □

   b) $\{r \in \mathbb{Q} : r^2 < 2\}$

   Solution. Note that $\{r \in \mathbb{Q} : r^2 < 2\} = \{r \in \mathbb{Q} : -\sqrt{2} < r < \sqrt{2}\}$, so

   $$\overline{\{r \in \mathbb{Q} : r^2 < 2\}} = [-\sqrt{2}, \sqrt{2}],$$

   since by density of $\mathbb{Q}$ every $x \in [-\sqrt{2}, \sqrt{2}]$ is a limit point of $\{r \in \mathbb{Q} : r^2 < 2\}$. □

3. Determine if the following series converge or diverge. Be sure to justify your answers.

   a) $\sum_{n=1}^{\infty} \dfrac{n^4}{2^n}$

   Solution. Since $\frac{n+1}{n} \to 1$,

   $$\begin{aligned} \overline{\lim_{n\to\infty}} \left| \frac{(n+1)^4/2^{n+1}}{n^4/2^n} \right| &= \overline{\lim_{n\to\infty}} \left| \left(\frac{n+1}{n}\right)^4 \frac{2^n}{2^{n+1}} \right| \\ &= \frac{1}{2} \overline{\lim_{n\to\infty}} \left| \left(\frac{n+1}{n}\right)^4 \right| \\ &= \frac{1}{2} < 1, \end{aligned}$$

   so the series converges absolutely by the ratio test. □

   b) $\sum_{n=1}^{\infty} \dfrac{2^n}{n!}$

Solution. The series converges absolutely by the ratio test, since

$$\varlimsup_{n\to\infty}\left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \varlimsup_{n\to\infty}\left|\frac{2}{n+1}\right| = 0 < 1.$$

$\square$

c) $\sum_{n=1}^{\infty} \dfrac{n!}{n^4+3}$

Solution. Note that for all $n \geq 1$

$$\frac{n!}{n^4+3} \geq \frac{n!}{n^4+3n^4} = \frac{n!}{4n^4} \geq 0.$$

The series $\sum_{n=1}^{\infty} \frac{n!}{4n^4}$ diverges by the ratio test since

$$\varlimsup_{n\to\infty}\left|\frac{(n+1)!/[4(n+1)^4]}{n!/[4n^4]}\right| = \varlimsup_{n\to\infty}\left|\left(\frac{n}{n+1}\right)^4 (n+1)\right| = \infty > 1.$$

Then $\sum_{n=1}^{\infty} \frac{n!}{n^4+3}$ diverges by the comparison test. $\square$

d) $\sum_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2}$

Solution. Note that for all $n \geq 1$,

$$\left|\frac{\cos^2 n}{n^2}\right| \leq \frac{1}{n^2}.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the integral test, the series $\sum_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2}$ converges absolutely by the comparison test. $\square$

e) $\sum_{n=2}^{\infty} \dfrac{1}{[n+(-1)^n]^2}$

Solution. Note that for all $n \geq 2$

$$\left|\frac{1}{[n+(-1)^n]^2}\right| \leq \frac{1}{(n-1)^2}.$$

Since the series $\sum_{n=2}^{\infty} \dfrac{1}{(n-1)^2}$ converges by the integral test, the series $\sum_{n=2}^{\infty} \dfrac{1}{[n+(-1)^n]^2}$ converges absolutely by the comparison test. $\square$

f) $\sum_{n=1}^{\infty} \dfrac{n!}{n^n}$

Solution. Note that

$$\left|\frac{(n+1)!/(n+1)^{(n+1)}}{n!/n^n}\right| = \left(\frac{n}{n+1}\right)^n = e^{\frac{\ln(n/(n+1))}{(1/n)}}.$$

By L'hôpital's rule

$$\lim_{n\to\infty} \frac{\ln(n/(n+1))}{(1/n)} = \lim_{n\to\infty} \frac{\frac{n+1}{n} \cdot \frac{1}{(n+1)^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty} -\frac{n}{n+1} = -1,$$

so

$$\overline{\lim_{n\to\infty}} \left| \frac{(n+1)!/(n+1)^{(n+1)}}{n!/n^n} \right| = e^{-1} < 1.$$

Therefore, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges absolutely by the ratio test. □

g) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$

Solution. Note that for all $n \geq 2$

$$\frac{1}{\sqrt{n}\ln n} \geq \frac{1}{n\ln n} \geq 0,$$

and

$$\int_2^{\infty} \frac{1}{x\ln x}\, dx = \ln\ln x|_2^{\infty} = \infty.$$

Hence, the series $\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ diverges by the integral test, which implies that $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ diverges by the comparison test. □

h) $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)}$

Solution. Note that

$$\int_4^{\infty} \frac{1}{x\ln x\ln\ln x}\, dx = \ln\ln\ln x|_4^{\infty} = \infty.$$

Hence, the series $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)}$ diverges by the integral test. □

i) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

Solution. Using integration by parts with $u = \ln x$ and $dv = (1/x^2)dx$, we get

$$\int_2^{\infty} \frac{\ln x}{x^2}\, dx = -\left.\frac{\ln x}{x}\right|_2^{\infty} - \int_2^{\infty} -\frac{1}{x^2}\, dx = \frac{\ln 2}{2} - \left.\frac{1}{x}\right|_2^{\infty} = \frac{\ln 2}{2} + \frac{1}{2}.$$

Since the integral converges, the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges by the integral test. □

4. Prove the following: If $\sum_{n=1}^{\infty} |a_n|$ converges and $(b_n)_n$ is a bounded sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution. Let $M > 0$ be such that $|b_n| \leq M$ for all $n \geq 1$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $m > n \geq N$ implies

$$\sum_{k=n+1}^{m} |a_k| < \frac{\varepsilon}{M}.$$

Then for $m > n \geq N$, we have

$$\left| \sum_{k=n+1}^{m} a_k b_k \right| \leq \sum_{k=n+1}^{m} |a_k||b_k| \leq \sum_{k=n+1}^{m} |a_k| M < M \cdot \frac{\varepsilon}{M} = \varepsilon.$$

Hence, by the Cauchy criterion, $\sum_{n=1}^{\infty} a_n b_n$ converges. □

5. Prove the following: If $\sum_{n=1}^{\infty} a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum_{n=1}^{\infty} a_n^p$ converges.

Solution. Since $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$. Choose $N_1 \in \mathbb{N}$ such that $0 \leq a_n < 1$ for all $n \geq N_1$. Then

$$n \geq N_1 \implies 0 \leq a_n^p \leq a_n < 1.$$

Let $\varepsilon > 0$. Choose $N_2 \in \mathbb{N}$ such that

$$m > n \geq N_2 \implies \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Then for $m > n \geq \max\{N_1, N_2\}$

$$\left| \sum_{k=n+1}^{m} a_k^p \right| \leq \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon.$$

Hence, $\sum_{n=1}^{\infty} a_n^p$ converges by the Cauchy criterion. □

6. Prove the following: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series of nonnegative numbers, then $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges. Hint: Show that $\sqrt{a_n b_n} \leq a_n + b_n$ for all $n \geq 1$.

Solution. Since $a_n, b_n \geq 0$ for all $n \geq 1$,

$$a_n b_n \leq 2a_n b_n \leq a_n^2 + 2a_n b_n + b_n^2 = (a_n + b_n)^2,$$

which implies

$$\sqrt{a_n b_n} \leq a_n + b_n.$$

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, $\sum_{n=1}^{\infty} (a_n + b_n)$ converges. Therefore, $\sum_{n=1}^{\infty} \sqrt{a_n b_n}$ converges by the comparison test. □