Topics in Basic Analysis: Homework 5

1. Suppose that the limits $L_1 = \lim_{x\to a} f_1(x)$ and $L_2 = \lim_{x\to a} f_2(x)$ exists.

   a) Prove that if $\exists c < a < d$ such that $f_1(x) \le f_2(x)$ for all $x \in (c, d) \setminus a$, then $L_1 \le L_2$.

   b) Is it true that if $f_1(x) < f_2(x)$ for all $x \in (c, d) \setminus a$, then $L_1 < L_2$?

2. Let $f : (a, b) \mapsto \mathbb{R}$ be continuous. Prove that if $f(r) = 0$ for all $r \in \mathbb{Q} \cap (a, b)$, then $f(x) = 0$ for all $x \in (a, b)$.

3. Let $f, g : (a, b) \mapsto \mathbb{R}$ be continuous. Prove that if $f(r) = g(r)$ for all $r \in \mathbb{Q} \cap (a, b)$, then $f(x) = g(x)$ for all $x \in (a, b)$.

4. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be defined by
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
   Show that f is not continuous at any $x \in \mathbb{R}$.

5. Let $h : \mathbb{R} \mapsto \mathbb{R}$ be defined by
$$h(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
   Prove that h is continuous at $x = 0$ only.

6. Let $f, g : [a, b] \mapsto \mathbb{R}$ be continuous function such that $f(a) \ge g(a)$ and $f(b) \le g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

7. Use Q6, to show that if $f : [0, 1] \mapsto [0, 1]$ is continuous, then f has a fixed point, i.e. $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

8. Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

9. Determine if the following functions are uniformly continuous. Be sure to justify your answers.

   a) $f(x) = x^3$ on $(0, 1)$

   b) $f(x) = x^3$ on $\mathbb{R}$

   c) $f(x) = \sin(1/x^2)$ on $(0, 1]$

   d) $f(x) = x^2 \sin(1/x)$ on $(0, 1]$

10. Prove that if $f : S \subseteq \mathbb{R} \mapsto \mathbb{R}$ is uniformly continuous and S is a bounded set, then f is bounded on S. Hint: Assume not and use the Bolzano-Weierstrass theorem and the the fact that for a uniformly continuous functions, $f(x_n)_n$ is Cauchy whenever $(x_n)_n \subset S$ is Cauchy.

11. Prove that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.