Topics in Basic Analysis: Homework 5 Solutions

1. Suppose that the limits $L_1 = \lim_{x\to a} f_1(x)$ and $L_2 = \lim_{x\to a} f_2(x)$ exists.

   a) Prove that if $\exists c < a < d$ such that $f_1(x) \le f_2(x)$ for all $x \in (c,d) \setminus a$, then $L_1 \le L_2$.

   Solution. Suppose $\exists c, d \in \mathbb{R}$ such that $c < a < d$ and $f_1(x) \le f_2(x)$ for all $x \in (c,d) \setminus \{a\}$.

   Method 1: $(\varepsilon - \delta)$ Suppose that $L_1 > L_2$. Then $L_1 - L_2 > 0$. Since $L_1 = \lim_{x\to a} f_1(x)$ and $L_2 = \lim_{x\to a} f_2(x)$, $\lim_{x\to a}(f_1(x) - f_2(x)) = L_1 - L_2$. Let $\varepsilon = (L_1 - L_2)/2$. Then $\exists\delta > 0$ such that for all $x \in (c,d) \setminus \{a\}$

   $$0 < |x-a| < \delta \implies |f_1(x) - f_2(x) - (L_1 - L_2)| < \frac{L_1 - L_2}{2}.$$

   This implies that for $x \in (c,d) \setminus \{a\}$ such that $0 < |x-a| < \delta$

   $$0 < \frac{L_1 - L_2}{2} < f_1(x) - f_2(x) \implies f_2(x) < f_1(x),$$

   a contradiction with $f_1(x) \le f_2(x)$, $\forall x \in (c,d) \setminus \{a\}$. Thus, $L_1 \le L_2$.

   Method 2: (Sequential characterization of limits) Let $(x_n)_n \subset (c,d) \setminus \{a\}$ such that $x_n \to a$. Then $L_1 = \lim_{x\to a} f_1(x)$ and $L_2 = \lim_{x\to a} f_2(x)$ implies that

   $$f_1(x_n) \to L_1 \text{ and } f_2(x_n) \to L_2.$$

   Since $f_2(x_n) - f_1(x_n) \ge 0$ for all $n \ge 1$,

   $$f_2(x_n) - f_1(x_n) \to L_2 - L_1 \implies L_2 - L_1 \ge 0.$$

   □

   b) Is it true that if $f_1(x) < f_2(x)$ for all $x \in (c,d) \setminus a$, then $L_1 < L_2$?

   Solution. No. As with the case of sequences, this is not true. Consider the functions

   $$f_1(x) = 0 \text{ and } f(x) = x$$

   with $a = 0$. Then clearly

   $$f_1(x) < f_2(x),\ \forall x \in (-1,1) \setminus \{0\}$$

   but

   $$\lim_{x\to 0} f_1(x) = \lim_{x\to 0} f_2(x) = 0.$$

   □

2. Let $f : (a,b) \mapsto \mathbb{R}$ be continuous. Prove that if $f(r) = 0$ for all $r \in \mathbb{Q} \cap (a,b)$, then $f(x) = 0$ for all $x \in (a,b)$.

Solution. Suppose $f : (a, b) \mapsto \mathbb{R}$ is continuous, $f(r) = 0$ for all $s \in \mathbb{Q} \cap (a, b)$, and that $f(x_0) \neq 0$ for some $x_0 \in (a, b)$. WLOG suppose that $f(x_0) > 0$. Let $\varepsilon = f(x_0)/2 > 0$. By continuity of f at x_0, $\exists \delta > 0$ such that for $x \in (a, b)$

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Then for all $x \in (x_0 - \delta, x_0 + \delta)$, we have

$$0 < \frac{f(x_0)}{2} < f(x),$$

but by density of $\mathbb{Q}$, $\exists r \in (x_0 - \delta, x_0 + \delta) \cap \mathbb{Q}$ and $f(r) = 0$, a contradiction. Thus $f(x) = 0$ for all $x \in (a, b)$. □

3. Let $f, g : (a, b) \mapsto \mathbb{R}$ be continuous. Prove that if $f(r) = g(r)$ for all $r \in \mathbb{Q} \cap (a, b)$, then $f(x) = g(x)$ for all $x \in (a, b)$.

   Solution. Let $h(x) = f(x) - g(x)$. Then $h : (a, b) \mapsto \mathbb{R}$ is continuous and $h(r) = 0$ for all $r \in \mathbb{Q} \cap (a, b)$, so by Q3, $h(x) = 0$ for all $x \in (a, b)$. Hence $f(x) = g(x)$ for all $x \in (a, b)$. □

4. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Show that f is not continuous at any $x \in \mathbb{R}$.

   Solution. Proof 1: $(\varepsilon - \delta)$ Let $x_0 \in \mathbb{R}$ and suppose that f is continuous at x_0. Then $x_0 \in \mathbb{Q}$ or $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Suppose that $x_0 \in \mathbb{Q}$. Let $\varepsilon = \frac{1}{2}$. Then by continuity, $\exists \delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{1}{2} \iff \frac{1}{2} < f(x) < \frac{3}{2}.$$

   Since the irrationals are dense in $\mathbb{R}$, $\exists$ an irrational number $t \in (x_0 - \delta, x_0 + \delta)$ but $f(t) = 0 < 1/2$ a contradiction. The argument is similar if instead $x_0 \in \mathbb{R} \setminus \mathbb{Q}$.

   Proof 2: (Sequential characterization of continuity) Suppose that h is continuous at $x_0 \in \mathbb{R}$. Again, we consider two cases, $x_0 \in \mathbb{Q}$ or $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Suppose that $x_0 \in \mathbb{Q}$. Since f is continuous at x_0, we must have $\forall (x_n)_n \subset \mathbb{R}$

$$x_n \to x_0 \implies f(x_n) \to f(x_0).$$

   By density of the irrationals, $\exists (x_n)_n \subset \mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x_0$, but $f(x_n) = 0$ for all $n \geq 1$ and $f(x_n) = 0 \to 0 \neq 1 = f(x_0)$, a contradiction. Thus f is not continuous at any $x \in \mathbb{Q}$. A similar argument shows that f is not continuous at any $x \in \mathbb{R} \setminus \mathbb{Q}$. □

5. Let $h : \mathbb{R} \mapsto \mathbb{R}$ be defined by

$$h(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Prove that h is continuous at $x = 0$ only.

Solution. Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Then for $x \in \mathbb{R}$

$$|x - 0| < \delta \implies |h(x) - h(0)| = |h(x)| \leq |x| < \delta = \varepsilon.$$

Thus, h is continuous at $x = 0$.

Suppose that h is continuous at $x_0 \neq 0$. We follow a similar argument to Q4. Suppose that $x_0 \in \mathbb{Q}$. By density of the irrationals, we can construct a sequence $(x_n)_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x_0$, but $h(x_n) = 0 \to 0 \neq x_0 = h(x_0)$, a contradiction. Thus, h cannot be continuous at x_0. Similar argument shows that h is not continuous at any $0 \neq x_0 \in \mathbb{R} \setminus \mathbb{Q}$. □

6. Let $f, g : [a, b] \mapsto \mathbb{R}$ be continuous function such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

   Solution. Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a, b]$, $h(a) = f(a) - g(a) \geq 0$ and $h(b) = f(b) - g(b) \leq 0$. If $h(a) = 0$ or $h(b) = 0$, then we are done. If $h(a) > 0$ and $h(b) < 0$, then by the intermediate value theorem, $\exists x_0 \in (a, b)$ such that $0 = h(x_0) = f(x_0) - g(x_0)$. □

7. Use Q6, to show that if $f : [0, 1] \mapsto [0, 1]$ is continuous, then f has a fixed point, i.e. $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$.

   Solution. Note that $f(0) \geq 0$ and $f(1) \leq 1$. Take $g(x) = x$. Then $f(0) \geq g(0)$, $f(1) \leq g(1)$, and f and g are continuous on $[0, 1]$. Then by Q6, $\exists x_0 \in [0, 1]$ such that $f(x_0) = g(x_0) = x_0$, i.e. f has a fixed point. □

8. Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

   Solution. Note $h(x) = \cos x - x$ is continuous on $[0, \pi/2]$, and

   $$h(0) = \cos(0) - 0 = 1 \quad \text{and} \quad h(\pi/2) = \cos(\pi/2) - \pi/2 = -\pi/2.$$

   Since $h(\pi/2) < 0 < h(0)$, by the intermediate value theorem, $\exists x_0 \in (0, \pi/2)$ such that $0 = h(x_0) = \cos x_0 - x_0$. □

9. Determine if the following functions are uniformly continuous. Be sure to justify your answers.

   a) $f(x) = x^3$ on $(0, 1)$

   Solution. Yes. Note that $f(x) = x^3$ is continuous on $[0, 1]$ and $[0, 1]$ is a compact set, so f is uniformly continuous on $[0, 1]$, which implies that f is uniformly continuous on $(0, 1)$. □

   b) $f(x) = x^3$ on $\mathbb{R}$

   Solution. No. Let $x_n = n + \frac{1}{n}$ and $y_n = n$. Then $(x_n)_n, (y_n)_n \subset \mathbb{R}$ and $|x_n - y_n| = \frac{1}{n} \to 0$, but

   $$|f(x_n) - f(y_n)| = \left| n^3 + 3n + \frac{3}{n} + \frac{1}{n^3} - n^3 \right| \to \infty,$$

   so f is not uniformly continuous on $\mathbb{R}$. □

   c) $f(x) = \sin(1/x^2)$ on $(0, 1]$

Solution. No. Let $x_n = \sqrt{\frac{2}{n\pi}}$. Then $(x_n)_n \subset (0,1]$. Since $x_n \to 0$, $(x_n)_n$ is a Cauchy sequence, but

$$(f(x_n))_n = \left\{\sin\left(\frac{n\pi}{2}\right)\right\}_n = \{1, 0, -1, 0, 1, \ldots\},$$

which does not converge and hence is not Cauchy. Thus, f is not uniformly continuous. □

d) $f(x) = x^2 \sin(1/x)$ on $(0, 1]$

Solution. Yes. Note that $\lim_{x\to 0} f(x) = 0$. If we consider

$$h(x) = \begin{cases} f(x), & x \neq 0 \\ 0, & x = 0 \end{cases},$$

then h is continuous on $\mathbb{R}$ and $h(x) = f(x)$ for all $x \in (0,1]$. In particular, h is continuous on $[0,1]$ and $[0,1]$ is compact, so h is uniformly continuous on $[0,1]$. Since h and f agree on $(0,1] \subset [0,1]$, f is uniformly continuous on $(0,1]$. □

10. Prove that if $f : S \subseteq \mathbb{R} \mapsto \mathbb{R}$ is uniformly continuous and S is a bounded set, then f is bounded on S. Hint: Assume not and use the Bolzano-Weierstrass theorem and the the fact that for a uniformly continuous functions, $f(x_n)_n$ is Cauchy whenever $(x_n)_n \subset S$ is Cauchy.

Solution. Suppose that f is not bounded on S. Let $(y_n)_n$ be an unbounded sequence in $f(S) = \{y \in \mathbb{R} | y = f(x) \text{ for some } x \in S\}$ such that $|y_n| \geq n$ for all $n \geq 1$. Then for each $n \in \mathbb{N}$, $y_n = f(x_n)$ for some $x_n \in S$. Since S is a bounded set and $(x_n)_n \subset S$, $(x_n)_n$ is a bounded sequence. By the Bolzano-Weierstrass theorem, $(x_n)_n$ has a convergent subsequence $(x_{n_k})_k$. Then $(x_{n_k})_k$ is also a Cauchy sequence. Since f is uniformly continuous and $y_{n_k} = f(x_{n_k})$, we must also have the $(y_{n_k})_k$ is Cauchy, and so $(y_{n_k})_k$ is also bounded. Let $M > 0$ be such that $|y_{n_k}| \leq M$ for all $k \geq 1$. Choose $N \in \mathbb{N}$ such that $N > M$. Then for $k \geq N$, $n_k \geq k \geq N$, so $|y_{n_k}| \geq n_k \geq N > M$, a contradiction. Thus, f must be bounded on S. □

11. Prove that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

Solution. Note that $|f'(t)| = |\cos t| \leq 1$ for all $x \in \mathbb{R}$. Since f is continuous and differentiable on $\mathbb{R}$ and f' is bounded, f is Lipschitz on $\mathbb{R}$. Hence, f is uniformly continuous □