Topics in Basic Analysis: Homework 6

1. Find the radius of convergence of the following power series, and determine whether or not they converge at each endpoint if $0 < R < \infty$.

   a) $\sum_{n=0}^{\infty} n^2 x^n$

   b) $\sum_{n=0}^{\infty} \dfrac{n^3}{3^n} x^n$

   c) $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{n^2 \cdot 4^n} x^n$

2. Prove that $f_n(x) = \dfrac{1+\cos^2(nx)}{\sqrt{n}}$ converges uniformly to 0 on $\mathbb{R}$.

3. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0,1]$.

   a) Does the sequence $\{f_n\}$ converge pointwise to some function f on $[0,1]$? If so, then find f.

   b) Does $\{f_n\}$ converge uniformly on $[0,1]$? Justify your answer.

4. Prove that if $f_n \to f$ uniformly on S, and $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ unformly on S.

5. Show that if $f_n \to f$ uniformly on S, and $g_n \to g$ uniformly on S, then it need not be true that $f_n g_n \to fg$ uniformly on S with the following example.

   a) Show that $f_n(x) = x$ converges uniformly to $f(x) = x$ on $\mathbb{R}$ and $g_n(x) = \frac{1}{n}$ converges uniformly to $g(x) = 0$ on $\mathbb{R}$.

   b) Show that $f_n g_n$ does not converge uniformly to fg on $\mathbb{R}$.

6. Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$ that converge uniformly to f on $[a,b]$. Show that if $(x_n)_n \subset [a,b]$ and $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$. Is this true, if $f_n \to f$ pointwise but not uniformly?

7. Consider the series $\sum_{n=0}^{\infty} \dfrac{x^n}{1+x^n}$

   a) Show that the series converges for all $x \in [0,1)$.

   b) Show that the series converges uniformly on $[0,a]$ for all $0 < a < 1$.

   c) Does the series converge uniformly on $[0,1)$? Explain.