Topics in Basic Analysis: Homework 6 Solutions

1. Find the radius of convergence of the following power series, and determine whether or not they converge at each endpoint if $0 < R < \infty$.

   a) $\sum_{n=0}^{\infty} n^2 x^n$

   Solution. Note that
   $$\overline{\lim_{n\to\infty}} |n^2|^{1/n} = 1,$$
   so the radius of convergence is $R = 1$. Thus, the series converges absolutely for all $x \in (-1,1)$. When $x = 1$, the series does not converge since
   $$\lim_{n\to\infty} n^2 = \infty \neq 0.$$
   Similarly, the series does not converge when $x = -1$ since
   $$\lim_{n\to\infty} n^2(-1)^n \neq 0.$$
   □

   b) $\sum_{n=0}^{\infty} \frac{n^3}{3^n} x^n$

   Solution. Note that
   $$\overline{\lim_{n\to\infty}} \left|\frac{n^3}{3^n}\right|^{1/n} = \overline{\lim_{n\to\infty}} \frac{(n^{1/n})^3}{3} = \frac{1}{3},$$
   so the radius of convergence is $R = 3$. Thus, the series converges absolutely for all $x \in (-3,3)$. At $x = 3$, the series diverges, since
   $$\lim_{n\to\infty} \frac{n^3}{3^n} \cdot 3^n = \lim_{n\to\infty} n^3 = \infty \neq 0.$$
   Similarly, the series diverges at $x = -3$, since
   $$\lim_{n\to\infty} \frac{n^3}{3^n}(-3)^n = \lim_{n\to\infty} (-1)^n n^3 \neq 0.$$
   □

   c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 \cdot 4^n} x^n$

   Solution. Note that
   $$\overline{\lim_{n\to\infty}} \left|\frac{(-1)^n}{n^2 \cdot 4^n}\right|^{1/n} = \overline{\lim_{n\to\infty}} \frac{1}{4(n^{1/n})^2} = \frac{1}{4},$$
   so the radius of convergence is $R = 4$. At $x = 4$, we have the alternating series
   $$\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2},$$
   which converges by the alternating series test, and at $x = -4$ we have the series
   $$\sum_{n=0}^{\infty} \frac{1}{n^2}$$
   which converges by the integral test. □

2. Prove that $f_n(x) = \dfrac{1+\cos^2(nx)}{\sqrt{n}}$ converges uniformly to 0 on $\mathbb{R}$.

   Solution. For all $x \in \mathbb{R}$, we have
   $$|f_n(x)| \leq \frac{1+1}{\sqrt{n}} = \frac{2}{\sqrt{n}}.$$
   Thus,
   $$\sup_{x\in\mathbb{R}} |f_n(x)| \leq \frac{2}{\sqrt{n}} \to 0,$$
   so $f_n \to 0$ uniformly on $\mathbb{R}$. □

3. Let $f_n(x) = (x - \frac{1}{n})^2$ for $x \in [0,1]$.

   a) Does the sequence $\{f_n\}$ converge pointwise to some function f on $[0,1]$? If so, then find f.

      Solution. For each $x \in [0.1]$,
      $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (x - \frac{1}{n})^2 = x^2 := f(x).$$
      Thus, $f_n \to f$ pointwise on $[0,1]$. □

   b) Does $\{f_n\}$ converge uniformly on $[0,1]$? Justify your answer.

      Solution. Yes. Note that
      $$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |(x - \frac{1}{n})^2 - x^2| = \sup_{x\in[0,1]} \left| -\frac{2x}{n} + \frac{1}{n^2} \right| \leq \frac{2}{n} + \frac{1}{n^2} \to 0.$$
      □

4. Prove that if $f_n \to f$ uniformly on S, and $g_n \to g$ uniformly on S, then $f_n + g_n \to f + g$ unformly on S.

   Solution. Method 1: (ε-Definition). Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ such that $n \geq N$ implies
   $$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ and } |g_n(x) - g(x)| < \frac{\varepsilon}{2}, \ \forall x \in S.$$
   Then for $n \geq N$
   $$\begin{aligned} |(f_n(x) + g_n(x)) - (f(x) + g(x))| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\ &\leq |f_n(x) - f(x)| + |g_n(x) + g(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$
   for all $x \in S$.

   Method 2: Recall that $f_n \to f$ and $g_n \to g$ uniformly on S if and only if
   $$\sup_{x\in S} |f_n(x) - f(x)| \to 0 \text{ and } \sup_{x\in S} |g_n(x) - g(x)| \to 0.$$
   Then
   $$\sup_{x\in S} |(f_n(x) + g_n(x)) - (f(x) + g(x))| \leq \sup_{x\in S} |f_n(x) - f(x)| + \sup_{x\in S} |g_n(x) - g(x)| \to 0,$$
   so $f_n + g_n \to f + g$ uniformly on S. □

5. Show that if $f_n \to f$ uniformly on S, and $g_n \to g$ uniformly on S, then it need not be true that $f_n g_n \to fg$ uniformly on S with the following example.

   a) Show that $f_n(x) = x$ converges uniformly to $f(x) = x$ on $\mathbb{R}$ and $g_n(x) = \frac{1}{n}$ converges uniformly to $g(x) = 0$ on $\mathbb{R}$.

   Solution. Note that
$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} |x - x| = 0 \to 0,$$
   and
$$\sup_{x\in\mathbb{R}} |g_n(x) - g(x)| = \sup_{x\in\mathbb{R}} \frac{1}{n} \to 0.$$
   □

   b) Show that $f_n g_n$ does not converge uniformly to fg on $\mathbb{R}$.

   Solution. Even though $f_n \to f$ and $g_n \to g$ uniformly on $\mathbb{R}$, note that
$$\sup_{x\in\mathbb{R}} |f_n(x)g_n(x) - f(x)g(x)| = \sup_{x\in\mathbb{R}} \frac{|x|}{n} = \infty \not\to 0,$$
   so $f_n g_n$ does not converge uniformly to fg in $\mathbb{R}$. □

6. Let $\{f_n\}$ be a sequence of continuous functions on $[a, b]$ that converge uniformly to f on $[a, b]$. Show that if $(x_n)_n \subset [a, b]$ and $x_n \to x$, then $\lim_{n\to\infty} f_n(x_n) = f(x)$. Is this true, if $f_n \to f$ pointwise but not uniformly?

   Solution. Since f_n is continuous for each n and $f_n \to f$ uniformly on $[a, b]$, f is also continuous. Let $(x_n)_n$ be a sequence in $[a, b]$ such that $x_n \to x$. Then
$$\begin{aligned} |f_n(x_n) - f(x)| &= |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \\ &\le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &\le \sup_{t\in[a,b]} |f_n(t) - f(t)| + |f(x_n) - f(x)| \\ &\to 0. \end{aligned}$$
   Note that the first time in the second to last line converges to 0 because $f_n \to f$ uniformly on $[a, b]$, and the second term converges to 0 by the continuity of f. □

7. Consider the series $\sum_{n=0}^{\infty} \dfrac{x^n}{1 + x^n}$

   a) Show that the series converges for all $x \in [0, 1)$.

   Solution. Note that for each $x \in [0, 1)$,
$$\frac{x^n}{1 + x^n} \le x^n,\ \forall n \ge 0.$$
   Since $\sum_{n=0}^{\infty} x^n$ (geometric series) converges for $x \in [0, 1)$, $\sum_{n=0}^{\infty} \dfrac{x^n}{1 + x^n}$ converges for $x \in [0, 1)$ by the comparison test. □

b) Show that the series converges uniformly on $[0, a]$ for all $0 < a < 1$.

Solution. Let $0 < a < 1$. Note that $f(x) = \dfrac{x^n}{1+x^n}$ is increasing in x on $[0, 1)$. To see this, note that

$$f'(x) = \frac{nx^{n-1}}{1+x^n} - \frac{nx^{2n-1}}{(1+x^n)^2} = \frac{nx^{n-1}}{(1+x^n)^2} > 0, \ \forall x \in (0, a).$$

Thus, for each $n \geq 0$,

$$\left|\frac{x^n}{1+x^n}\right| \leq \frac{a^n}{1+a^n} \leq a^n.$$

Since $\sum_{n=0}^{\infty} a^n < \infty$, $\sum_{n=0}^{\infty} \dfrac{x^n}{1+x^n}$ converges uniformly on $[0, a]$ by the M-test. □

c) Does the series converge uniformly on $[0, 1)$? Explain.

Solution. No. Note that

$$\sup_{x\in[0,1)} \left|\sum_{n=0}^{m} \frac{x^n}{1+x^n} - \sum_{n=0}^{\infty} \frac{x^n}{1+x^n}\right| = \sup_{x\in[0,1)} \sum_{n=m+1}^{\infty} \frac{x^n}{1+x^n} \geq \sup_{x\in[0,1)} \sum_{n=m+1}^{\infty} x^n = \sup_{x\in[0,1)} \frac{x^{m+1}}{1-x} = \infty.$$

□