# Topics in Basic Analysis: Homework 7

1. Let
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$
   a) Using differentiation formulas from calculus, show that $f$ is differentiable for all $x \neq 0$ and find a formula for $f'(x)$, $x \neq 0$.
   b) Use the definition of derivative to show that $f$ is differentiable at $x = 0$ and that $f'(0) = 0$.
   c) Show that $f'$ is not continuous at $x = 0$.

2. Let
$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$
   a) Prove that $f$ is not differentiable at $x = 0$.
   b) Is $f$ continuous at $x = 0$? Justify your answer.

3. Suppose that $f$ is differentiable at $x = a$. Prove the following statements.
   a) $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$
   b) $\lim_{h\to 0} \dfrac{f(a+h) - f(a-h)}{2h} = f'(a)$

4. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

5. Suppose that $f$ is differentiable on $\mathbb{R}$ and that $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$.
   a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.
   b) Show that $f'(x) = 0$ for some $x \in (1, 2)$.

6. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that $f$ is a constant functions.

7. Show that $x < \tan x$ for all $x \in (0, \pi/2)$.

8. Show that $x/\sin x$ is a strictly increasing function on $(0, \pi/2)$.

9. Show that $x \leq \frac{\pi}{2} \sin x$ for $x \in [0.\pi/2]$.

10. Suppose that $f$ is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2$ for all $x \in \mathbb{R}$, and that $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all $x \geq 0$.

11. Find the Taylor series for $\cos x$ centered at 0, and prove that it converges to $\cos x$ for all $x \in \mathbb{R}$.