Topics in Basic Analysis: Homework 7 Solutions

1. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

   a) Using differentiation formulas from calculus, show that f is differentiable for all $x \neq 0$ and find a formula for $f'(x)$, $x \neq 0$.

   Solution. For $x \neq 0$, note that

$$\frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2},$$

   so

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x},$$

   by applying the product, quotient and chain rules. □

   b) Use the definition of derivative to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

   Solution.

$$|f'(0)| = \lim_{x \to 0} \left| \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} \right| = \lim_{x \to 0} |x \sin \frac{1}{x}| \leq \lim_{x \to 0} |x| = 0.$$

   □

   c) Show that f' is not continuous at $x = 0$.

   Solution. Let $x_n = \frac{1}{2n\pi}$, $n \geq 1$. Then $x_n \to 0$, but

$$f'(x_n) = 2(2n\pi)\sin(2n\pi) - \cos(2n\pi) = -1 \to -1 \neq 0 = f'(0).$$

   Hence, f' is not continuous at $x = 0$. □

2. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

   a) Prove that f is not differentiable at $x = 0$.

   Solution. Note that

$$\lim_{x \to 0} \frac{x \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$$

   does not exist. To see this, consider the sequence $x_n = 2/[(2n+1)\pi]$, then

$$\sin \frac{1}{x_n} = \sin \frac{(2n+1)\pi}{2} = (-1)^n,$$

   which does not have a limit. □

   b) Is f continuous at $x = 0$? Justify your answer.

Solution. Yes. Note that $|\sin x| \leq 1$ for all $x \in \mathbb{R}$ implies that

$$\lim_{x\to 0} |f(x) - f(0)| = \lim_{x\to 0} |x \sin \frac{1}{x}| \leq \lim_{x\to 0} |x| = 0,$$

by the squeeze theorem. □

3. Suppose that f is differentiable at $x = a$. Prove the following statements.

a) $\lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

Solution. Let $\varepsilon > 0$. Since f is differentiable at a, $\exists \delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon.$$

Then for $0 < |h| < \delta$, $0 < |a + h - a| = |h| < \delta$. Thus $0 < |h| < \delta$ implies

$$\left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| = \left| \frac{f(a+h) - f(a)}{a + h - a} - f'(a) \right| < \varepsilon.$$

□

b) $\lim_{h\to 0} \dfrac{f(a+h) - f(a-h)}{2h} = f'(a)$

Solution. Method 1: Let $\varepsilon > 0$. Since f is differentiable at a, $\exists \delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon.$$

Then for $0 < |h| < \delta$, $0 < |a + h - a| < \delta$ and $0 < |a - h - a| < \delta$. Thus $0 < |h| < \delta$ implies

$$\begin{aligned} \left| \frac{f(a+h) - f(a-h)}{2h} - f'(a) \right| &= \left| \frac{f(a+h) - f(a) + f(a) - f(a-h)}{2h} - f'(a) \right| \\ &\leq \frac{1}{2} \left| \frac{f(a+h) - f(a)}{h} - f'(a) \right| + \frac{1}{2} \left| \frac{f(a-h) - f(a)}{-h} - f'(a) \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Method 2: Note that for $h \neq 0$,

$$\begin{aligned} \frac{f(a+h) - f(a-h)}{2h} &= \frac{1}{2} \left[\frac{f(a+h) - f(a) + f(a) - f(a-h)}{h} \right] \\ &= \frac{1}{2} \left[\frac{f(a+h) - f(a)}{h} + \frac{f(a + (-h)) - f(a)}{-h} \right] \\ &\to \frac{1}{2} [f'(a) + f'(a)] \end{aligned}$$

by part a) and the fact that $h \to 0 \iff -h \to 0$.

□

4. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

Solution. Since $\cos x$ is continuous and differentiable on $\mathbb{R}$, the MVT guarantees that for every $x, y \in \mathbb{R}$, $\exists t$ between x and y such that

$$\cos x - \cos y = -\sin t(x - y).$$

Since $|\sin t| \leq 1$ for all $t \in \mathbb{R}$, we obtain

$$|\cos x - \cos y| = |\sin t||x - y| \leq |x - y|.$$

Since x and y were arbitrary, we are done. □

5. Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 0$, $f(1) = 1$, and $f(2) = 1$.

   a) Show that $f'(x) = \frac{1}{2}$ for some $x \in (0, 2)$.

   Solution. Since f is differentiable on $\mathbb{R}$, it is also continuous on $\mathbb{R}$. Since

   $$\frac{f(2) - f(0)}{2 - 0} = \frac{1}{2},$$

   $\exists x \in (0, 2)$ such that $f'(x) = \frac{1}{2}$, by the MVT. □

   b) Show that $f'(x) = 0$ for some $x \in (1, 2)$.

   Solution. Note that f is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Since

   $$\frac{f(2) - f(1)}{2 - 1} = 0,$$

   $\exists x \in (1, 2)$ such that $f'(x) = 0$ by the MVT. (You can also apply Rolle's theorem here as well.) □

6. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant functions.

Solution. Note that for all $x \neq y$,

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|.$$

Thus,

$$\lim_{x \to y}\left|\frac{f(x) - f(y)}{x - y}\right| \leq \lim_{x \to y}|x - y| = 0,$$

which implies that $f'(y) = 0$ for all $y \in \mathbb{R}$. Then by a result from class, f is constant on $\mathbb{R}$. □

7. Show that $x < \tan x$ for all $x \in (0, \pi/2)$.

Solution. Let $h(x) = \tan x - x$. Note that $h(0) = 0$ and

$$h'(x) = \sec^2 x - 1 > 0, \ \forall x \in (0, \pi/2).$$

This implies that h is strictly increasing on $[0, \pi/2)$, so that for all $x \in (0, \pi/2)$, we have

$$h(x) > 0 \implies \tan x > x.$$

□

8. Show that $x/\sin x$ is a strictly increasing function on $(0, \pi/2)$.

Solution. For $x \in (0, \pi/2)$,

$$\frac{d}{dx}\frac{x}{\sin x} = \frac{1}{\sin x} - \frac{x\cos x}{\sin^2 x} = \frac{\tan x - x}{\cos x \sin^2 x} > 0,$$

since $\tan x - x > 0$ (by Q7), $\cos x > 0$, and $\sin^2 x > 0$ for all $x \in (0, \pi/2)$. This implies that $\dfrac{x}{\sin x}$ is strictly increasing on $(0, \pi/2)$. □

9. Show that $x \leq \frac{\pi}{2}\sin x$ for $x \in [0, \pi/2]$.

Solution. The inequality clearly holds at $x = 0$. By Q8, $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2]$. Since

$$\frac{\pi}{2}\frac{\sin(\pi/2)}{\pi/2} = 1,$$

we have that $\frac{\pi}{2}\frac{\sin x}{x} \geq 1$ for all $x \in (0, \pi/2]$, and the result follows. □

10. Suppose that f is differentiable on $\mathbb{R}$, $1 \leq f'(x) \leq 2$ for all $x \in \mathbb{R}$, and that $f(0) = 0$. Prove that $x \leq f(x) \leq 2x$ for all $x \geq 0$.

Solution. The result is clear for $x - 0$. Let $x > 0$. By the MVT, $\exists$ and $t \in (0, x)$ such that

$$f'(t) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Since $1 \leq f'(t) \leq 2$ for all $t \in \mathbb{R}$, we have

$$1 \leq f'(t) = \frac{f(x)}{x} \implies x \leq f(x)$$

and

$$2 \geq f'(t) = \frac{f(x)}{x} \implies 2x \geq f(x).$$

□

11. Find the Taylor series for $\cos x$ centered at 0, and prove that it converges to $\cos x$ for all $x \in \mathbb{R}$.

Solution. Let $f(x) = \cos x$. Then

$$f^{(1)}(x) = -\sin x,\ f^{(2)}(x) = -\cos x,\ f^{(3)}(x) = \sin x,\ f^{(4)}(x) = \cos x, \ldots$$

Since $\sin 0 = 0$ and $\cos 0 = 1$, we have the Taylor series expansion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}.$$

By Taylor's theorem, for each $x \neq 0$ and for each $n \geq 1$, $\exists \xi$ between 0 and x such that

$$\left|\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}\right| = \left|\frac{f^{(2n+1)}(\xi)}{(2n+1)!}x^{2n+1}\right| \leq \frac{|x|^{2n+1}}{(2n+1)!} \to 0,$$

since $|f^{(n)}(\xi)| \leq 1$ for all $n \geq 1$ and for all $\xi \in \mathbb{R}$. To see why $|x|^n/n! \to 0$, consider the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$. Then for $x \neq 0$

$$\limsup_{n\to\infty} \left| \frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} \right| = \limsup_{n\to\infty} \frac{|x|}{n+1} = 0.$$

Thus, by the ratio test, the series $\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$ converges for all $x \neq 0$ (and clearly for $x = 0$), so that

$$\frac{|x|^n}{n!} \to 0.$$

□