Topics in Basic Analysis: Homework 8

Throughout, let $F : [a, b] \mapsto \mathbb{R}$ be a monotonically increasing function with $< -\infty < F(a) < F(b) < \infty$, and by $h \in \mathscr{R}(F, [a, b])$, we mean that h is Riemann-Stieltjes integrable with respect to F over $[a, b]$.

1. Let h be a bounded function. Suppose that there exists a sequence of upper and lower Darboux-Stieltjes sums for h with respect to F over $[a, b]$ such that $\lim_{n\to\infty}(U_n - L_n) = 0$. Show that $h \in \mathscr{R}(F, [a, b])$ and that $\int_a^b h\ dF = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.

2. Let $h \in \mathscr{R}(F, [a, b])$, and suppose that g is a function on $[a, b]$ such that $h(x) = g(x)$ except at finitely many points in $[a, b]$. Show that $\int_a^b h\ dF = \int_a^b g\ dF$.

3. Show that if $h \in \mathscr{R}(F, [a, b])$, then $h \in \mathscr{R}(F, [c, d])$ for every $[c, d] \subset [a, b]$.

4. Show that if $h(x) \geq 0$ for all x, h is continuous, and $h \in \mathscr{R}([a, b])$ with $\int_a^b h\ \ dt = 0$, then $h(x) = 0$ for all $x \in [a, b]$.

5. Let $h, g \in \mathscr{R}(F, [a, b])$.

   a) Show that $\min\{h, g\} = \frac{1}{2}[(h + g) - |h - g|]$ and that $\max\{h, g\} = -\min\{-h, -g\}$.

   b) Use part a), to show that $\max\{h, g\}, \min\{h, g\} \in \mathscr{R}(F, [a, b])$.

6. Suppose that h and g are continuous functions on $[a, b]$ such that $g(x) \geq 0$ for all $x \in [a, b]$. Prove that there exists and $x \in [a, b]$ such that

$$\int_a^b h(t)g(t)\ dF(t) = h(x) \int_a^b g(t)\ dF(t).$$

7. Use Q6, to prove the intermediate value theorem for integrals: If h is continuous on $[a, b]$, then there exists and $x \in [a, b]$ such that

$$h(x) = \frac{1}{F(b) - F(a)} \int_a^b h\ dF.$$

8. Calculate the following limits:

   a) $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2}\ dt$

   b) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\ dt$