Topics in Basic Analysis: Homework 8 Solutions

Throughout, let $F : [a, b] \mapsto \mathbb{R}$ be a monotonically increasing function with $< -\infty < F(a) < F(b) < \infty$, and by $h \in \mathscr{R}(F, [a, b])$, we mean that h is Riemann-Stieltjes integrable with respect to F over $[a, b]$.

1. Let h be a bounded function. Suppose that there exists a sequence of upper and lower Darboux-Stieltjes sums for h with respect to F over $[a, b]$ such that $\lim_{n\to\infty}(U_n - L_n) = 0$. Show that $h \in \mathscr{R}(F, [a, b])$ and that $\int_a^b h\, dF = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.

   Solution. Suppose h is bounded on $[a, b]$ and suppose there exists sequences $(P_n)_n$ and $(Q_n)_n$ of partitions of $[a, b]$ such that
$$U(h, P_n, F) - L(h, Q_n, F) \to 0.$$
   Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that
$$U(h, P_N, F) - L(h, Q_N, F) < \varepsilon,$$
   and let $P = P_N \cup Q_N$ be the refinement of P_N with Q_N. Then
$$U(h, P, F) - L(h, P, F) \leq U(h, P_N, F) - L(h, Q_N, F) < \varepsilon,$$
   so $h \in \mathscr{R}(F, [a, b])$ by the Cauchy criterion for integrability. Next, note that for all $n \geq 1$
$$L(h, Q_n, F) \leq \int_a^b h\, dF \leq U(h, P_n, F).$$
   This implies that
$$0 \leq U(h, P_n, F) - \int_a^b h\, dF \leq U(h, P_n, F) - L(h, Q_n, F) \to 0,$$
   and
$$0 \leq \int_a^b h\, dF - L(h, Q_n, F) \leq U(h, P_n, F) - L(h, Q_n, F) \to 0.$$
   □

2. Let $h \in \mathscr{R}(F, [a, b])$, and suppose that g is a function on $[a, b]$ such that $h(x) = g(x)$ except at finitely many points in $[a, b]$. Show that $\int_a^b h\, dF = \int_a^b g\, dF$.

   Solution. Suppose $h \in \mathscr{R}(F, [a, b])$, and that $g(x) = f(x)$ for all $x \in [a, b]$ except at a finite number of points $\{x_1, \dots, x_n\}$. Choose a partition P of $[a, b]$ such that
$$U(h, P, F) - L(h, P, F) < \frac{\varepsilon}{2}.$$
   Let $d = \max\{|g(x_i) - h(x_i)| : i = 1, \dots, m\}$ and let $M = \sup_{x\in[a,b]} |h(x)| < \infty$, since h is integrable. Let $Q = \{t_k\}_{k=0}^m$ be a refinement of P such that $\text{mesh}_F(Q) < \frac{\varepsilon}{8n(M+d)}$. WLOG assume that each interval $[t_{j-1}, t_j], j = 1, \dots, m$, contains at most one of $x_k, k = 1, 2, \dots, n$ (otherwise, we can construct a refinement of P where this is true). Let $A = \{j \in \{1, 2, \dots, m\} : x_k \in [t_{j-1}, t_j]\}$, and note that $|A| \leq 2n$, since each x_k is contained in at most two intervals $[t_{j-1}, t_j]$

(It is either a point in the partition P, in which case x_k lies in exactly two such intervals, or it lies within only one interval). Then

$$\begin{aligned}
U(g,Q,F) - L(g,Q,F) &= U(h,Q,F) - L(h,Q,F) \\
&\quad + \sum_{j\in A}[M(g,[t_{j-1},t_j]) - m(g,[t_{j-1},t_j])]\Delta F_j \\
&\quad - \sum_{j\in A}[\underbrace{M(h,[t_{j-1},t_j]) - m(h,[t_{j-1},t_j])}_{\geq 0}]\Delta F_j \\
&< \frac{\varepsilon}{2} + \sum_{j\in A}[(2(M+d))]\Delta F_j \\
&< \frac{\varepsilon}{2} + 2n\cdot 2(M+d)\cdot \frac{\varepsilon}{8n(M+d)} \\
&= \varepsilon.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
|U(h,Q,F) - U(g,Q,F)| &= \left|\sum_{j\in A}[M(h,[t_{j-1},t_j]) - M(g,[t_{j-1},t_j])]\Delta F_j\right| \\
&\leq \sum_{j\in A}(2M+d)\cdot \frac{\varepsilon}{8n(M+d)} \\
&\leq \varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
\left|\int_a^b h\,dF - \int_a^b g\,dF\right| &\leq U(h,Q,F) - \int_a^b h\,dF + |U(h,Q,F) - U(g,Q,F)| \\
&\quad + U(g,Q,P) - \int_a^b g\,dF \\
&< \frac{\varepsilon}{2} + \varepsilon + \varepsilon.
\end{aligned}$$

□

3. Show that if $h \in \mathscr{R}(F,[a,b])$, then $h \in \mathscr{R}(F,[c,d])$ for every $[c,d] \subset [a,b]$.

Solution. Let P be a partition of $[a,b]$ such that

$$U(h,P,F) - L(h,P,F) < \varepsilon.$$

Consider the refinement of P given by $Q = P \cup \{c,d\} = \{t_k\}_{k=0}^n$. Suppose that $c = t_{k_1}$ and $d = t_{k_2}$. Then $Q^* = \{t_k\}_{k=k_1}^{k_2}$ is a partition of $[c,d]$ and

$$\begin{aligned}
U(h,Q^*,F) - L(h,Q^*,F) &= \sum_{j=k_1}^{k_2}[M(h,[t_{j-1},t_j]) - m(h,[t_{j-1},t_j])\Delta F_j] \\
&\leq \sum_{j=1}^{n}[M(h,[t_{j-1},t_j]) - m(h,[t_{j-1},t_j])\Delta F_j]
\end{aligned}$$

$$\begin{aligned} &= U(h,Q,F) - L(h,Q,F) \\ &\le U(h,P,F) - L(h,P,F) \\ &< \varepsilon. \end{aligned}$$

Thus, $h \in \mathscr{R}(F,[c,d])$ by the Cauchy criterion for integrability. □

4. Show that if $h(x) \geq 0$ for all x, h is continuous, and $h \in \mathscr{R}([a,b])$ with $\int_a^b h \ dt = 0$, then $h(x) = 0$ for all $x \in [a,b]$.

   Solution. Suppose that $h(x_0) > 0$. Since h is continuous at x_0, we can choose a $\delta > 0$ such that $h(x) > h(x_0)/2$ for all $x \in [x_0 - \delta, x_0 + \delta]$. Let $\varepsilon = 2\delta \cdot h(x_0)/2$. Since

$$0 = \int_a^b h \, dt = \inf_P U(h,P),$$

   we can choose a partition P of $[a,b]$ such that $U(h,P) < \varepsilon$. Consider that refinement of P given by $Q = P \cup \{x_0 - \delta, x_0 + \delta\}$. Then

$$\varepsilon > U(h,P,F) \geq U(H,Q) \geq \frac{h(x_0)}{2} \cdot 2\delta = \varepsilon,$$

   a contradiction. Thus, $h(x) = 0$ for all $x \in [a,b]$.

   Alternatively, we can argue as follows using the construction above, additivity, and the order property of the integral. Note that

$$\int_a^b h \, dt = \underbrace{\int_a^{x_0-\delta} h \, dt}_{\geq 0} + \int_{x_0-\delta}^{x_0+\delta} \underbrace{h}_{\geq h(x_0)/2} \, dt + \underbrace{\int_{x_0+\delta}^{b} h \, dt}_{\geq 0} \geq \frac{h(x_0)}{2} \cdot 2\delta > 0,$$

   again contradicting that $\int_a^b h \, dt = 0$. □

5. Let $h, g \in \mathscr{R}(F,[a,b])$.

   a) Show that $\min\{h,g\} = \frac{1}{2}[(h+g) - |h-g|]$ and that $\max\{h,g\} = -\min\{-h,-g\}$.

   Solution. If $h(x) < g(x)$, then $|h(x) - g(x)| = g(x) - h(x)$ and

$$\frac{1}{2}[(h(x)+g(x)) - |h(x)-g(x)|] = \frac{1}{2}[h(x)+g(x)-(g(x)-h(x))] = \frac{1}{2}\cdot 2h(x) = h(x) = \min\{h(x),g(x)\}.$$

   Similarly, if $h(x) \geq g(x)$, then $|h(x) - g(x)| = h(x) - g(x)$ and

$$\frac{1}{2}[(h(x)+g(x)) - |h(x)-g(x)|] = \frac{1}{2}[h(x)+g(x)-(h(x)-g(x))] = \frac{1}{2}\cdot 2g(x) = g(x) = \min\{h(x),g(x)\}.$$

   Note that $-\min\{-h,-g\} = -\frac{1}{2}[(-h+-g) - |-h--g|] = \frac{1}{2}[(h+g) + |h-g|]$. A similar argument to the one above shows that this is equal to $\max\{h,g\}$. □

   b) Use part a), to show that $\max\{h,g\}, \min\{h,g\} \in \mathscr{R}(F,[a,b])$.

   Solution. Since $h, g \in \mathscr{R}(F,[a,b])$, we have $h+g, h-g \in \mathscr{R}(F,[a,b])$, which also implies that $|h-g| \in \mathscr{R}(F,[a,b])$. The result then follows by linearity of the integral. □

6. Suppose that h and g are continuous functions on $[a,b]$ such that $g(x) \geq 0$ for all $x \in [a,b]$. Prove that there exists and $x \in [a,b]$ such that

$$\int_a^b h(t)g(t)\, dF(t) = h(x) \int_a^b g(t)\, dF(t).$$

Solution. Since h is continuous on $[a,b]$ it is bounded, and by the EVT, it attains its max and min on $[a,b]$. That is $\exists x_l, x_u \in [a,b]$ such that $h(x_l) \leq h(x) \leq h(x_u)$ for all $x \in [a,b]$. Then $h(x_l)g(t) \leq h(x)g(x) \leq h(x_u)g(x)$ for all $x \in [a,b]$. Since h, g are continuous, $h, g \in \mathscr{R}(F, [a,b])$, and by the order and linearity properties for integrals

$$h(x_l) \int_a^b g\, dF \leq \int_a^b hg\, dF \leq h(x_u) \int_a^b g\, dF.$$

If $\int_a^b g\, dF = 0$, then $g(x) = 0$ for all $x \in [a,b]$, which implies that $\int_a^b hg\, dF = 0$, and the result follows. If $\int_a^b g\, dF > 0$, then we have

$$h(x_l) \leq \frac{1}{\int_a^b g\, dF} \int_a^b hg\, dF \leq h(x_u).$$

Since h is continuous, the IVT implies that there exists an x between x_l and x_u such that

$$h(x) = \frac{1}{\int_a^b g\, dF} \int_a^b hg\, dF.$$

□

7. Use Q6, to prove the intermediate value theorem for integrals: If h is continuous on $[a,b]$, then there exists and $x \in [a,b]$ such that

$$h(x) = \frac{1}{F(b) - F(a)} \int_a^b h\, dF.$$

Solution. Let $g(x) = 1$ for all $x \in [a,b]$. Then g is continuous on $[a,b]$ and $\int_a^b g\, dF = F(b) - F(a)$. By Q6, there exists and $x \in [a,b]$ such that

$$h(x) = \frac{1}{\int_a^b g\, dF} \int_a^b hg\, dF = \frac{1}{F(b) - F(a)} \int_a^b h\, dF.$$

□

8. Calculate the following limits:

a) $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt$

Solution. Let $F(x) = \int_{-1}^x e^{t^2}\, dt$. Since e^{t^2} is continuous on $[-1,1]$, F is also continuous on $[-1,1]$ and differentiable at 0 with $F'(0) = e^{0^2} = 1$. Note that

$$\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\, dt = \lim_{x \to 0} \frac{F(x) - F(0)}{x - 0} = F'(0) = 1.$$

□

b) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt$

Solution. Let $F(x) = \int_0^x e^{t^2}\, dt$. Similar to part a), F is differentiable at 3 with $F'(3) = e^9$, so

$$\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\, dt = \lim_{h\to 0} \frac{F(3+h) - F(3)}{h} = F'(3) = e^{3^2} = e^9.$$

□