The Helly-Bray Theorem

One question of interest in probability and statistics is under what conditions does

$$EX_n \to EX,$$

or for what class of functions g does

$$Eg(X_n) \to Eg(X).$$

We study one such set of conditions here with the Helly-Bray theorem.

Definition 1. A **subdistribution function** (sdf) F is a real-valued function defined on $\mathbb{R}$ satisfying

(i) F is nondecreasing.

(ii) F is right-continuous on $\mathbb{R}$.

(iii) $F(-\infty) = \lim_{t\to-\infty} F(t) \geq 0$ and $F(\infty) = \lim_{t\to\infty} F(t) \leq 1$.

Note 1. Every cumulative distribution function (cdf) is an sdf, and an sdf is a cdf if we have equality in (iii), i.e. both $F(-\infty) = 0$ and $F(\infty) = 1$.

Definition 2. For a sdf, F, let $C(F)$ denote the set of (finite) continuity points of F, i.e.

$$C(F) = \{x \in \mathbb{R} : F(x) = F(x-)\}.$$

Note 2. Recall that a bounded, monotonic function can have at most countably many jump discontinuities, so $[C(F)]^c$ = the set of discontinuity points is countable. This implies that $\forall x \in \mathbb{R}$, $\exists (x_n)_n \subset C(F)$ and $(y_n)_n \subset C(F)$ with $x_n \uparrow x$ and $y_n \downarrow x$.

Proof. Suppose that for some $x \in \mathbb{R}$, there does not exist such sequences $(x_n)_n$ and $(y_n)_n$. This implies that there exists a $\delta > 0$ such that

$$(x-\delta, x+\delta) \cap C(F) = \emptyset,$$

so then

$$\underbrace{(x-\delta, x+\delta)}_{\text{uncountable}} \subset \underbrace{[C(F)]^c}_{\text{countable}}.$$

□

We now introduce two modes of convergence. I do want to note here that the terminology used here is old and not widely used this way anymore, but I want to make a distinction between convergence to a cdf vs an sdf.

Definition 3. Let $\{F_n, n \geq 1\}$ be a sequence of cdf's, and let F be an sdf. We say that F_n **converges weakly** to F, written $F_n \xrightarrow{w} F$, if

$$F_n(t) \to F(t),\ \forall t \in C(F).$$

If F is also a cdf and $F_n \xrightarrow{w} F$, then we say that F_n **converges completely** to F, written $F_n \xrightarrow{c} F$.

Example 1. Let $X_n \overset{iid}{\sim} U(-n, n)$. Then

$$F_n(x) = \begin{cases} 0, & x < -n \\ \dfrac{1}{2n}x + \dfrac{1}{2}, & -n \le x \le n \\ 1, & x > n \end{cases}$$

Then $F_n(t) \to \frac{1}{2}$ for all $t \in \mathbb{R}$. Thus, $F_n \overset{w}{\to} F \equiv \frac{1}{2}$, but $F_n \overset{c}{\not\to} F$, since F is not a cdf.

Definition 4. If $\{X_n, n \ge 1\}$ are R.V.'s with cdf's $\{F_n, n \ge 1\}$, and X is a R.V. with cdf F, then we say that X_n **converges in distribution to** X, written $X_n \overset{d}{\to} X$ if $F_n \overset{c}{\to} F$.

Example 2. Does convergence in distribution have similar properties to limits? Similar to other probabilistic convergence modes like convergence in probability? Below are some examples to get you thinking about convergence in distribution, but we leave the solutions for a later course.

1. (Uniqueness of limits?) Show that if $X_n \overset{d}{\to} X_F$ and $X_n \overset{d}{\to} X_G$, then $F(x) = G(x)$, $\forall x \in \mathbb{R}$. (Hint: Consider points in $C(F) \cap C(G)$ first.)

2. (Is the rate for convergence in distribution always $\sqrt{n}$ like in the CLT?) Let $X_n \overset{iid}{\sim} U(0, 1)$. Find a cdf F, so that
$$n(1 - X_{(n)}) \overset{d}{\to} X_F,$$
where $X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}$.

3. (Do we have additivity of limits for convergence in distribution?) Is it true that if $X_n \overset{d}{\to} X$ and $Y_n \overset{d}{\to} Y$, then $X_n + Y_n \overset{d}{\to} X + Y$? In general, this is not true, but if we add independence between X_n and Y_n, then we can achieve such a result. To why this is not true in general, consider $X_n = Z \sim N(0, 1)$ and $Y_n = -Z \sim N(0, 1)$.

Theorem 1 (The Finite Helly-Bray Theorem)**.** *Let $\{F_n, n \ge 1\}$ be cdf's such that $F_n \overset{w}{\to} F$ and let $a, b \in C(F)$, where $a < b$. Then for every real continuous function g on $[a, b]$*

$$\lim_{n\to\infty} \int_a^b g\, dF_n = \int_a^b g\, dF.$$

Proof. Let $\varepsilon > 0$. Since $[a, b]$ is compact, g is uniformly continuous on $[a, b]$, so we can choose a $\delta > 0$, such that

$$a \le x, y \le b,\ |x - y| < \delta \implies |g(x) - g(y)| < \varepsilon.$$

Choose $x_i \in C(F)$, $0 \le i \le k$, such that

$$a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b \text{ and } \max_{0\le i\le k-1}\{x_{i+1} - x_i\} < \delta.$$

(Note that I can always choose $x_i \in C(F)$) by Note 2.) Let $M < \infty$ be such that $|g(x)| \le M$ for all $x \in [a, b]$. Then

$$H_n \overset{(let)}{=} \int_a^b g\, dF_n - \int_a^b g\, dF = \sum_{i=0}^{k-1} \left(\int_{x_i}^{x_{i+1}} g\, dF_n - \int_{x_i}^{x_{i+1}} g\, dF \right)$$

$$= \sum_{i=0}^{k-1} \left(\int_{x_i}^{x_{i+1}} [g(x) - g(x_i)] \, dF_n(x) + \int_{x_i}^{x_{i+1}} g(x_i) \, dF_n(x) \right.$$
$$\left. - \int_{x_i}^{x_{i+1}} g(x_i) \, dF(x) + \int_{x_i}^{x_{i+1}} [g(x_i) - g(x)] \, dF(x) \right)$$

and so

$$\begin{aligned}
|H_n| &\leq \sum_{i=0}^{k-1} \left(\int_{x_i}^{x_{i+1}} |g(x) - g(x_i)| \, dF_n(x) + \int_{x_i}^{x_{i+1}} |g(x_i) - g(x)| \, dF(x) \right. \\
&\qquad \left. + |g(x_i)||F_n(x_{i+1}) - F_n(x_i) - F(x_{i+1})|F(x_i)| \right) \\
&\leq \varepsilon \sum_{i=0}^{k-1} (F_n(x_{i+1}) - F_n(x_i) + F(x_{i+1}) - F(x_i)) \\
&+ M \sum_{i=0}^{k-1} (|F_n(x_{i+1}) - F(x_{i+1})| + |F_n(x_i) - F(x_i)|) \\
&= \varepsilon(F_n(b) - F_n(a) + F(b) - F(a)) + o(1).
\end{aligned}$$

Thus, $\overline{\lim}_{n\to\infty} |H_n| \leq 2\varepsilon$, so $\lim_{n\to\infty} H_n = 0$. □

Theorem 2 (The Extended Helly-Bray Theorem)**.** *Let $F_n \xrightarrow{c} F$ and let g be a bounded continuous function on $\mathbb{R}$. Then*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g \, dF_n = \int_{-\infty}^{\infty} g \, dF.$$

Note 3. The Extended Helly-Bray theorem says that if $X_n \xrightarrow{d} X_F$, then

$$E[g(X_n)] \to E[g(X_F)]$$

for all bounded continuous functions g. The reverse implications turns out to be true as well, so this provides an equivalent definition of convergence in distribution, though we do not prove that here.

Proof. Let $M = \sup_{x\in\mathbb{R}} |g(x)| < \infty$. Let $a, b \in C(F)$ with $a < b$. Then

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} g \, dF_n - \int_{-\infty}^{\infty} g \, dF \right| \\
&\leq \left| \int_{-\infty}^{\infty} g \, dF_n - \int_{a}^{b} g \, dF_n \right| + \underbrace{\left| \int_{a}^{b} g \, dF_n - \int_{a}^{b} g \, dF \right|}_{\to 0 \text{ by Finite Helly-Bray}} + \left| \int_{a}^{b} g \, dF - \int_{-\infty}^{\infty} g \, dF \right| \\
&\leq M[F_n(a) - F_n(-\infty) + F_n(\infty) - F_n(b) + F(a) - F(-\infty) + F(\infty) - F(b)] + o(1) \\
&\overset{n\to\infty}{\to} M[F(a) + 1 - F(b) + F(a) + 1 - F(b)] \\
&\overset{a\to-\infty, b\to\infty}{\to} 0,
\end{aligned}$$

where $a \to -\infty$ and $b \to \infty$ along points in $C(F)$. □

Example 3. The following example shows that the Extended Helly-Bray theorem can fail if $F_n \xrightarrow{c} F$ is replaces by $F_n \xrightarrow{w} F$. Let $X_n \overset{iid}{\sim} U(-n, n)$. Then $F_n \xrightarrow{w} F \equiv \frac{1}{2}$. Let $g(x) = 1$ for all $x \in \mathbb{R}$. Then g is bounded and continuous on $\mathbb{R}$, but

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g \, dF_n = 1 \text{ while } \int_{-\infty}^{\infty} g \, dF = 0.$$

Example 4. The Extended Helly-Bray theorem can also fail if g is not bounded. Let $g(x) = x^2$ and let $\{X_n, n \geq 1\}$ be R.V.'s with

$$P(X_n = n) = \frac{1}{n^2} = 1 - P(X_n = 0).$$

Then $X_n \xrightarrow{P} X_F \equiv 0$, so that $X_n \xrightarrow{d} X_F$, but

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g \, dF_n = \lim_{n\to\infty} E[X_n^2] = 1 \quad \text{while} \quad \int_{-\infty}^{\infty} g \, dF = EX_F^2 = 0.$$