Inequalities

Definition 1. Let $-\infty \leq a < b \leq \infty$. A function $g : (a,b) \mapsto \mathbb{R}$ is said to be **convex** if for all $a < x_1 < x_2 < b$ and $0 \leq \lambda \leq 1$

$$g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda)g(x_2).$$

Note 1. Geometrically, a function is convex if the line segment between $(x_1, g(x_1))$ and $(x_2, g(x_2))$ does not go below the curve of $g(x)$ for $x \in (x_1, x_2)$.

Lemma 1. *If $g : (a,b) \mapsto \mathbb{R}$ is convex, then g is continuous on (a,b).*

Proof. Let $a < s < t < u < b$. Then

$$t = \frac{u-t}{u-s}s + \left(1 - \frac{u-t}{u-s}\right)u = \frac{u-t}{u-s}s + \frac{t-s}{u-s}u,$$

so by convexity, we have

$$g(t) \leq \frac{u-t}{u-s}g(s) + \frac{t-s}{u-s}g(u). \tag{1}$$

Now, fix s and u in (1). Then

$$\overline{\lim_{t \to s^+}}\, g(t) \leq g(s).$$

Next, fix t and s in (1). Then

$$\underline{\lim_{u \to t^+}}\, g(u) \geq g(t).$$

Hence, for any $x \in (a,b)$,

$$\overline{\lim_{t \to x^+}}\, g(t) \leq g(x) \leq \underline{\lim_{t \to x^+}}\, g(t) \implies \lim_{t \to x^+} g(t) = g(x).$$

If we fix t and u in (1), then

$$\underline{\lim_{s \to t^-}}\, g(s) \geq g(t).$$

If we fix s and u in (1), then

$$\overline{\lim_{t \to u^-}}\, g(t) \leq g(u).$$

Therefore, for any $x \in (a,b)$

$$\overline{\lim_{t \to x^-}}\, g(t) \leq g(x) \leq \underline{\lim_{t \to x^-}}\, g(t) \implies \lim_{t \to x^-} g(t) = g(x).$$

Together, theses imply that for all $x \in (a,b)$,

$$g(x) = g(x+) = g(x-), \text{ i.e. } g \text{ is continuous at } x.$$

$\square$

For a convex function $g : (a,b) \mapsto \mathbb{R}$, note that as in the prior proof, we have

$$g(t) \le \left(1 - \frac{t-s}{u-s}\right) g(s) + \frac{t-s}{u-s} g(u),$$

and so

$$\frac{g(t)-g(s)}{t-s} \le \frac{g(u)-g(s)}{u-s}, \; a < s < t < u < b.$$

Also note that

$$g(t) \le \frac{u-t}{u-s} g(s) + \left(1 - \frac{u-t}{u-s}\right) g(u),$$

and so

$$\frac{g(u)-g(s)}{u-s} \le \frac{g(u)-g(t)}{u-t}, \; a < s < t < u < b.$$

Hence

$$\frac{g(t)-g(s)}{t-s} \le \frac{g(u)-g(t)}{u-t}, \; a < s < t < u < b. \tag{2}$$

Theorem 1 (Jensen's Inequality). *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a convex function, and let X be a random variables such that $E|X| < \infty$. Then $Eg(X)$ exists and*

$$g(EX) \le Eg(X).$$

Proof. It follows from (2) that

$$M \stackrel{def}{=} \sup_{s<EX} \frac{g(EX)-g(s)}{EX-s} \le \frac{g(u)-g(EX)}{u-EX}, \; \forall u > EX. \tag{3}$$

Now (3) implies that

$$g(EX) - g(s) \le M(EX - s), \; \forall s < EX,$$

or, equivalently, that

$$g(s) - g(EX) \ge M(s - EX), \; \forall s < EX.$$

Clearly, $g(EX) - g(EX) \ge M(EX - EX)$. Also, (3) implies

$$g(u) - g(EX) \ge M(u - EX), \; \forall u > EX.$$

Hence,

$$g(x) - g(EX) \ge M(x - EX), \; \forall x \in \mathbb{R},$$

i.e.

$$g(x) \ge M(x - EX) + g(EX), \; \forall x \in \mathbb{R}.$$

Thus,

$$g(X) \ge M(X - EX) + g(EX).$$

It follows that $Eg(X)$ exists (possibly infinite), and by taking expectations of both sides we have

$$Eg(X) \ge 0 + g(EX) = g(EX).$$

□

Definition 2. For $0 < p < \infty$, the **p-norm** of a random variable X is defined by

$$\|X\|_p \stackrel{def}{=} \left(\int_{-\infty}^{\infty} |x|^p dF \right)^{1/p} (\leq \infty).$$

Theorem 2 (Liapounov Inequality). *Let X be a random variable, and let $0 < q < p < \infty$. Then*

$$(E|X|^q)^{1/q} \leq (E|X|^p)^{1/p},$$

i.e. $\|X\|_q \leq \|X\|_p$.

Proof. Let $r > 1$ and let Y be a random variable such that $E|Y| < \infty$. Then by Jensen's inequality, applied to the function $g(x) = x^r$ and the random variable $|Y|$, we have

$$(E|Y|)^r = g(E|Y|) \leq Eg(|Y|) = E|Y|^r. \qquad (\star)$$

This inequality trivially holds if $E|Y| = \infty$. In $(\star)$, replace Y by $|X|^q$ and replace r by $\frac{p}{q} > 1$ yielding

$$(E|X|^q)^{p/q} \leq E|X|^{q \cdot \frac{p}{q}} = E|X|^p,$$

which yields the conclusion. □

Theorem 3. *Let X, Y be random variables*

(i) (Hölder's inequality) Let $1 < p < \infty$ and $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (p and q are called conjugate indicies). Then

$$E|XY| \leq (E|X|^p)^{1/p} (E|Y|^q)^{1/q},$$

that is $\|XY\|_1 \leq \|X\|_p \|Y\|_q$.

(ii) (The Schwarz inequality). If $EX^2 < \infty$ and $EY^2 < \infty$, then $E|XY| < \infty$ and

$$|EXY| \leq E|XY| \leq \sqrt{EX^2 \cdot EY^2}.$$

(iii) (The Minkowski inequality) If $E|X|^p < \infty$ and $E|Y|^p < \infty$, where $p \geq 1$, then

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.$$

Proof. (i) If $\|X\|_p$ is 0 or ∞, or if $\|Y\|_q$ is 0 or ∞, then the inequality is clear. Otherwise, set

$$U = \frac{|X|}{\|X\|_p} \text{ and } V = \frac{|Y|}{\|Y\|_q}.$$

Note that $-\log t$ is a convex function on $(0, \infty)$ implying that if $a > 0$ and $b > 0$, then

$$-\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \leq -\frac{1}{p}\log a^p - \frac{1}{q}\log b^q = -\log ab.$$

Thus,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and so

$$UV \leq \frac{U^p}{p} + \frac{V^q}{q}.$$

Now,

$$EU^p = E\left(\frac{|X|^p}{\|X\|_p^p}\right) = 1$$

and similarly $EV^q = 1$, and so

$$EUV \leq \frac{1}{p} + \frac{1}{q} = 1,$$

i.e.

$$\frac{E|XY|}{\|X\|_p\|Y\|_q} \leq 1.$$

(ii) By noting that $\frac{1}{2} + \frac{1}{2} = 1$, the result follows by applying Hölder's inequality.

(iii) We omit the proof of this inequality, but note that this is the triangly inequality for the L_p space norm.

□